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An extension of Stern's congruence

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Abstract

Let $\{E_n\}$ be the Euler numbers. In the paper we determine $E_{2^mk+b}-E_b$ modulo 2^{m+7}, where k and m are positive integers and $b\in\{0,2,4,\ldots\}$.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers. The Euler numbers $\{E_n\}$ are given by

$$E_0=1,\quad E_{2n-1}=0,\quad \sum_{r=0}^{n}\binom{2n}{2r}E_{2r}=0\quad (n=1,2,3,\ldots).$$

The first few Euler numbers are shown below:

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ E_{10}=-50521,$$
$$E_{12}=2702765,\ E_{14}=-199360981,\ E_{16}=19391512145.$$

For $k,m\in\mathbb{N}$ and $b\in\{0,2,4,\ldots\}$, in 1875 Stern ([2]) proved the following congruence, which is now known as Stern's congruence:

$$E_{2^mk+b}\equiv E_b+2^mk\ (\mathrm{mod}\ 2^{m+1}). \tag{1.1}$$

There are many modern proofs of (1.1). See for example [1,3,5,6].

Let $b\in\{0,2,4,\ldots\}$ and $k,m\in\mathbb{N}$. In [4] the first author showed that

$$E_{2^mk+b}\equiv E_b+2^mk\ (\mathrm{mod}\ 2^{m+2})\quad \text{for}\quad m\ge 2, \tag{1.2}$$
$$E_{2^mk+b}\equiv E_b+5\cdot 2^mk\ (\mathrm{mod}\ 2^{m+3})\quad \text{for}\quad m\ge 3 \tag{1.3}$$

and

$$E_{2^mk+b}\equiv\begin{cases}E_b+5\cdot 2^mk\ (\mathrm{mod}\ 2^{m+4}) & \text{if } b\equiv 0,6\ (\mathrm{mod}\ 8),\\ E_b-3\cdot 2^mk\ (\mathrm{mod}\ 2^{m+4}) & \text{if } b\equiv 2,4\ (\mathrm{mod}\ 8)\end{cases}\quad \text{for}\quad m\ge 4. \tag{1.4}$$

From [4, Corollary 3.2] we know that for any nonnegative integer k and positive integer n,

$$\frac{3^{2k+1}+1}{4}E_{2k} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{2r+1}+1}{4}E_{2r} \pmod{2^{3n}}. \tag{1.5}$$

In the paper we use (1.5) to obtain a congruence for $E_{2^m k+b} - E_b$ modulo 2^{m+7}, where $k, m \in \mathbb{N}$ and $b \in \{0, 2, 4, \ldots\}$. In particular,

$$E_{2^m k+b} \equiv E_b + 2^m k(7(b+1)^2 - 18) \pmod{2^{m+7}} \quad \text{for} \quad m \geq 7. \tag{1.6}$$

Throughout the paper we use $\mathbb{Z}_2$ to denote the set of rational numbers whose denominator is odd.

2. The main result

Lemma 2.1. *Let $b \in \{0, 2, 4, \ldots\}$. Then*

$$E_b \equiv \begin{cases} 1 - 11b + 15b^2 + b^3 - b^4 \pmod{2^{10}} & \text{if } 4 \mid b, \\ 289 - 91b - 17b^2 - 7b^3 + b^4 \pmod{2^{10}} & \text{if } 4 \mid b-2. \end{cases}$$

Proof. By [4, Corollary 3.7],

$$E_b \equiv \begin{cases} \dfrac{1+172b-24b^2}{1-329b-74b^2-24b^3} \pmod{2^{10}} & \text{if } 4 \mid b, \\ \dfrac{-7-308(b-2)+40(b-2)^2}{7+111(b-2)+102(b-2)^2-24(b-2)^3} \pmod{2^{10}} & \text{if } 4 \mid b-2. \end{cases}$$

It is easy to see that

$$\begin{aligned}&(1-329b-74b^2-24b^3)(1-11b+15b^2+b^3-b^4)\\ &\equiv 1+172b-24b^2 \pmod{2^{10}}\end{aligned}$$

for $4 \mid b$, and that

$$\begin{aligned}&(7+111(b-2)+102(b-2)^2-24(b-2)^3)(289-91b-17b^2-7b^3+b^4)\\ &\equiv -7-308(b-2)+40(b-2)^2 \pmod{2^{10}}\end{aligned}$$

for $4 \mid b-2$. Thus the result follows.

Theorem 2.1. *Let $b \in \{0, 2, 4, \ldots\}$ and $k, m \in \mathbb{N}$. Then*

$$\begin{aligned}&E_{2^m k+b} - E_b\\ &\equiv \begin{cases} 2k(-(b-7)^2+38+2k(3b-1+2k)) \pmod{2^{m+7}} & \text{if } m=1 \text{ and } 2 \mid k, \\ 2k(-(b+1)^2+6+2k(3b-1+2k)) \\ \quad -16(b+(-1)^{\frac{b}{2}}) \pmod{2^{m+7}} & \text{if } m=1 \text{ and } 2 \nmid k, \\ 4k(7(b+1)^2-18 \\ \quad +12k(b+1+4((-1)^{\frac{b}{2}}-k))) \pmod{2^{m+7}} & \text{if } m=2, \\ 2^m k(7(b+1)^2-18+2^m k(7-b)) \pmod{2^{m+7}} & \text{if } m \geq 3. \end{cases}\end{aligned}$$

Proof. For $m = 1, 2, 3$, one can easily deduce the result from Lemma 2.1. Now we assume that $m \geq 4$. Set

$$f(k) = \frac{3^{2k+b+1}+1}{4}E_{2k+b} \quad \text{and} \quad F(k) = 2^{-3k}\sum_{r=0}^{k}\binom{k}{r}(-1)^r f(r).$$

From [4, Lemma 2.3] we know that $F(k) \in \mathbb{Z}_2$. By the binomial inversion formula we have $f(k) = \sum_{r=0}^{k}\binom{k}{r}(-2)^{3r}F(r)$ and so

$$\begin{aligned} f(2^{m-1}k) &= F(0) + \sum_{r=1}^{2^{m-1}k}\binom{2^{m-1}k}{r}(-2)^{3r}F(r) \\ &= F(0) + \sum_{r=1}^{2^{m-1}k}\frac{2^{m-1}k}{r}\binom{2^{m-1}k-1}{r-1}(-2)^{3r}F(r). \end{aligned}$$

For $r \geq 3$ we have $\frac{2^{r-2}}{r} \in \mathbb{Z}_2$ and $F(r) \in \mathbb{Z}_2$. Thus

$$\frac{2^{m-1}\cdot 2^{3r}F(r)}{r} = 2^{m+2r+1}\cdot\frac{2^{r-2}}{r}F(r) \equiv 0 \pmod{2^{m+7}}.$$

Hence,

(2.1)

$$\begin{aligned} &\frac{3^{2^m k+b+1}+1}{4}E_{2^m k+b} \\ &= f(2^{m-1}k) \equiv F(0) - 2^{m+2}kF(1) + 2^{m+4}k(2^{m-1}k-1)F(2) \\ &= \frac{3^{b+1}+1}{4}E_b - 2^{m-1}k(f(0)-f(1)) - 2^{m-2}k(f(0)-2f(1)+f(2)) + 2^{2m+3}k^2F(2) \\ &\equiv \frac{3^{b+1}+1}{4}E_b - 2^{m-2}k\,(3f(0)-4f(1)+f(2)) \pmod{2^{m+7}}. \end{aligned}$$

Putting $n = 3$ and $k = b/2$ in (1.5) we see that

$$\begin{aligned} \frac{3^{b+1}+1}{4}E_b &\equiv \sum_{r=0}^{2}(-1)^{2-r}\binom{b/2-1-r}{2-r}\binom{b/2}{r}\frac{3^{2r+1}+1}{4}E_{2r} \\ &= \frac{(\frac{b}{2}-1)(\frac{b}{2}-2)}{2} - \frac{b}{2}\frac{b}{2}(\frac{b}{2}-2)\cdot 7\cdot(-1) + \frac{\frac{b}{2}(\frac{b}{2}-1)}{2}\cdot 61\cdot 5 \\ &= 40b^2 - 84b + 1 \pmod{2^9}. \end{aligned}$$

Thus,

(2.2)

$$\begin{aligned} &3f(0) - 4f(1) + f(2) \\ &\equiv 3(40b^2-84b+1) - 4(40(b+2)^2 - 84(b+2)+1) + 40(b+4)^2 - 84(b+4) + 1 \\ &= 336 - 320b \equiv -176 - 64b \pmod{2^9}. \end{aligned}$$

For $r \geq 3$ we see that $4^{r-3}/r \in \mathbb{Z}_2$ and so

$$\binom{2^{m-2}k}{r}80^r = 2^{m+4+2r}\cdot\frac{4^{r-3}}{r}\cdot k\cdot 5^r\binom{2^{m-2}k-1}{r-1} \equiv 0 \pmod{2^{m+9}}.$$

Thus,

$$\begin{aligned} 3^{2^m k} &= (1+80)^{2^{m-2}k} = \sum_{r=0}^{2^{m-2}k} \binom{2^{m-2}k}{r} 80^r \\ &\equiv 1 + 5k\cdot 2^{m+2} + 25k \cdot 2^{m+5}(2^{m-2}k-1) \\ &= 1 - 195\cdot 2^{m+2}k + 25k^2\cdot 2^{2m+3} \\ &\equiv 1 + 61\cdot 2^{m+2}k + 2^{2m+3}k^2 \pmod{2^{m+9}}. \end{aligned}$$

This together with (2.1) and (2.2) yields

$$\begin{aligned} &\frac{3^{b+1}(1+61\cdot 2^{m+2}k + 2^{2m+3}k^2)+1}{4} E_{2^m k+b} \\ &\equiv \frac{3^{2^m k+b+1}+1}{4} E_{2^m k+b} \equiv \frac{3^{b+1}+1}{4} E_b + 2^{m-2}k(176+64b) \pmod{2^{m+7}}. \end{aligned}$$

Thus,

(2.3)

$$\frac{3^{b+1}+1}{4}(E_{2^m k+b} - E_b) \equiv 2^m k(2^{m+1}k - 55)3^b E_{2^m k+b} + 2^m k(44+16b) \pmod{2^{m+7}}.$$

By [3, Corollary 7.9] or Lemma 2.1,

$$E_b \equiv \begin{cases} 3b^2 - 11b + 1 \pmod{2^7} & \text{if} \quad 4 \mid b, \\ b^2 - 23b + 41 \pmod{2^7} & \text{if} \quad 4 \mid b-2. \end{cases} \tag{2.4}$$

Replacing b with $2^m k + b$ in (2.4) we get

$$E_{2^m k+b} \equiv \begin{cases} 3b^2 - 11b + 1 + 2^m k(6b-11) \pmod{2^7} & \text{if} \quad 4 \mid b, \\ b^2 - 23b + 41 + 2^m k(2b-23) \pmod{2^7} & \text{if} \quad 4 \mid b-2. \end{cases} \tag{2.5}$$

As

$$\begin{aligned} 3^b = (1+8)^{b/2} &= 1 + \binom{b/2}{1}8 + \binom{b/2}{2}8^2 + \cdots \equiv 8b^2 - 12b + 1 \\ &\equiv \begin{cases} -12b + 1 \pmod{2^7} & \text{if } 4 \mid b, \\ 20b - 31 \pmod{2^7} & \text{if } 4 \mid b-2, \end{cases} \end{aligned}$$

by (2.5) we have

$$3^b E_{2^m k+b} \equiv \begin{cases} (3b^2 - 11b + 1 + 2^m k(6b-11))(-12b+1) \pmod{2^7} & \text{if} \quad 4 \mid b, \\ (b^2 - 23b + 41 + 2^m k(2b-23))(20b-31) \pmod{2^7} & \text{if} \quad 4 \mid b-2. \end{cases}$$

If $4 \mid b$, then

$$(3b^2 - 11b + 1)(-12b+1) = -36b^3 + 135b^2 - 23b + 1 \equiv 7b^2 - 23b + 1 \pmod{2^7}$$

and

$$(6b-11)(-12b+1) \equiv -11 \equiv -3 \pmod{8}.$$

If $4 \mid b-2$, then $32 \mid (b-2)(b+2)$ and so

$$(b^2 - 23b + 41)(20b-31) = ((b-2)^2 - 19(b-2) - 1)(20(b+2) - 71)$$

$$\begin{aligned}&\equiv -20(b+2)-71(b^2-23b+41)\\&\equiv -7b^2+13b-7 \pmod{2^7}\end{aligned}$$

and

$$(2b-23)(20b-31)=(2(b-2)-19)(20b-31)\equiv 19\cdot 31\equiv -3 \pmod 8.$$

Thus

$$3^bE_{2^mk+b}\equiv\begin{cases}7b^2-23b+1-3\cdot 2^mk \pmod{2^7} & \text{if}\quad 4\mid b,\\ -7b^2+13b-7-3\cdot 2^mk \pmod{2^7} & \text{if}\quad 4\mid b-2.\end{cases}$$

Substituting this into (2.3) we obtain

$$\begin{aligned}&\frac{3^{b+1}+1}{4}(E_{2^mk+b}-E_b)\\&\equiv\begin{cases}2^mk(-b^2+b-11-2^mk) \pmod{2^{m+7}} & \text{if}\quad 4\mid b,\\ 2^mk(b^2+5b+45+3\cdot 2^mk) \pmod{2^{m+7}} & \text{if}\quad 4\mid b-2.\end{cases}\end{aligned}\tag{2.6}$$

It is easily seen that

$$\frac{3^{b+1}+1}{4}=\frac{3(1+8)^{b/2}+1}{4}\equiv\frac{3(1+\frac{b}{2}\cdot 8+\binom{b/2}{2}\cdot 8^2)+1}{4}=6b^2-9b+1 \pmod{2^7}.$$

If $4\mid b$, then $b^3=4^3(\frac{b}{4})^3\equiv 4^3\cdot\frac{b}{4}=16b \pmod{2^7}$. One can easily see that

$$(6b^2-9b+1)(3b^2-7b+1)\equiv 1 \pmod{2^7},\quad 3b^2-7b+1\equiv b+1 \pmod 8$$

and

$$(-b^2+b-11)(3b^2-7b+1)\equiv -b^2+14b-11 \pmod{2^7}.$$

Thus, by (2.6) we have

$$\begin{aligned}E_{2^mk+b}-E_b&\equiv 2^mk(-b^2+b-11-2^mk)(3b^2-7b+1)\\&\equiv 2^mk(-b^2+14b-11-(b+1)\cdot 2^mk)\\&\equiv 2^mk(7b^2+14b-11+2^mk(7-b)) \pmod{2^{m+7}}.\end{aligned}$$

If $4\mid b-2$, then $(b+2)^3\equiv 16(b+2) \pmod{2^7}$. One can easily see that

$$\begin{aligned}&(6b^2-9b+1)(-3(b+2)^2-7(b+2)+3)\equiv 1 \pmod{2^7},\\&-3(b+2)^2-7(b+2)+3\equiv b+5 \pmod 8\end{aligned}$$

and

$$\begin{aligned}&(b^2+5b+45)(-3(b+2)^2-7(b+2)+3)\\&=((b+2)^2+(b+2)+39)(-3(b+2)^2-7(b+2)+3)\\&\equiv -b^2+14b+21\equiv 7b^2+14b-11 \pmod{2^7}.\end{aligned}$$

Thus, by (2.6) we have

$$E_{2^mk+b}-E_b\equiv 2^mk(b^2+5b+45-2^{m+1}k)(-3(b+2)^2-7(b+2)+3)$$

$$\equiv 2^m k(7b^2+14b-11+2^m k(7-b)) \pmod{2^{m+7}}.$$

Note that $7b^2+14b-11 = 7(b+1)^2-18$. Combining all the above we prove the theorem.

Corollary 2.1. *Let $k, m \in \mathbb{N}$ with $m \geq 2$. Then*

$$E_{2^m k} \equiv \begin{cases} 4k(-48k^2+60k-11)+1 \pmod{2^{m+7}} & \text{if } m=2, \\ 2^m k(-11+7\cdot 2^m k)+1 \pmod{2^{m+7}} & \text{if } m \geq 3 \end{cases}$$

and

$$E_{2^m k+2} \equiv \begin{cases} 4k(-48k^2-12k+45)-1 \pmod{2^{m+7}} & \text{if } m=2, \\ 2^m k(45+5\cdot 2^m k)-1 \pmod{2^{m+7}} & \text{if } m \geq 3. \end{cases}$$

Corollary 2.2. *Let $b \in \{0,2,4,\ldots\}$ and $k, m \in \mathbb{N}$ with $m \geq 3$. Then*

$$E_{2^m k+b} \equiv E_b - 2^m k((b+1)^2+10+2^m k(b+1)) \pmod{2^{m+6}}.$$

Proof. Since $7(b+1)^2-18 \equiv -(b+1)^2-10 \pmod{2^6}$ and $7-b \equiv -b-1 \pmod 8$, the result follows from Theorem 2.1.

Corollary 2.3. *Let $b \in \{0,2,4,\ldots\}$ and $k, m \in \mathbb{N}$ with $m \geq 5$. Then*

$$E_{2^m k+b} \equiv \begin{cases} E_b - 11\cdot 2^m k \pmod{2^{m+5}} & \text{if } b \equiv 0, 14 \pmod{16}, \\ E_b + 13\cdot 2^m k \pmod{2^{m+5}} & \text{if } b \equiv 2, 12 \pmod{16}, \\ E_b - 3\cdot 2^m k \pmod{2^{m+5}} & \text{if } b \equiv 4, 10 \pmod{16}, \\ E_b + 5\cdot 2^m k \pmod{2^{m+5}} & \text{if } b \equiv 6, 8 \pmod{16}. \end{cases}$$

Proof. By Corollary 2.2 we have

$$E_{2^m k+b} \equiv E_b - 2^m k((b+1)^2+10) \pmod{2^{m+5}}.$$

Thus the result follows.

In conclusion we pose the following conjecture.

Conjecture 2.1. *Let $m, n \in \mathbb{N}$, $m \geq n$ and $b \in \{0,2,4,\ldots,2^{m+n-1}-2\}$. Then*

$$E_{2^m k+b} - E_b \equiv E_{2^m k+2^{m+n-1}-2-b} - E_{2^{m+n-1}-2-b} \pmod{2^{m+n}}.$$

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