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Congruences for sums involving Franel numbers

Zhi-Hong Sun

School of Mathematical Sciences
Huaiyin Normal University
Huaian, Jiangsu 223300, P.R. China
zhsun@hytc.edu.cn

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Let $\{f_n\}$ be the Franel numbers given by $f_n=\sum_{k=0}^n\binom nk^3$, and let $p>5$ be a prime. In this paper we mainly determine $\sum_{k=0}^{p-1}\binom{2k}k\frac{f_k}{m^k}\pmod p$ for $m=5,-16,16,32,-49,50,96$.

Keywords: Franel number; congruence.

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1. Introduction

In this paper we use the following notation. Let $\mathbb Z$ be the set of integers, and for a prime p let $\mathbb Z_p$ be the set of rational numbers whose denominator is not divisible by p. For positive integers a,b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly write that $n=ax^2+by^2$. Let $[x]$ be the greatest integer not exceeding x, and let $(\frac ap)$ be the Jacobi symbol.

In 1894 J. Franel [9] introduced the Franel numbers

$$f_n=\sum_{k=0}^n\binom nk^3\quad(n=0,1,2,\ldots)$$

and stated that

$$(n+1)^2f_{n+1}=(7n^2+7n+2)f_n+8n^2f_{n-1}\ (n\ge 1).$$

The first few Franel numbers are listed below:

$$f_0=1,\ f_1=2,\ f_2=10,\ f_3=56,\ f_4=346,\ f_5=2252,$$
$$f_6=15184,\ f_7=104960,\ f_8=739162,\ f_9=5280932.$$

Let p be an odd prime. In [10], Guo proved that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k}\equiv 0 \pmod p \quad \text{for} \quad p\equiv 3 \pmod 4)$$

and

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-16)^k}\binom{2k}{k}f_k\equiv(-1)^{\frac{p-1}{2}}p \pmod{p^3},$$

where the two congruences modulo p^2 were conjectured by the author. We note that $p\mid\binom{2k}{k}$ for $k=\frac{p+1}{2},\frac{p+3}{2},\ldots,p-1$. In [22,23], the author's brother Z.W. Sun investigated congruences for Franel numbers. In particular, he showed that for any odd prime p,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \pmod{p^2}.$$

By [15, Theorems 3.3 and 3.4],

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3).\end{cases}$$

Thus,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 3 \text{ and so } p=x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 2 \pmod 3).\end{cases}$$

For any nonnegative integer n let

$$\begin{aligned}A_n&=\sum_{k=0}^n\binom nk^2\binom{n+k}{k}^2,\quad a_n=\sum_{k=0}^n\binom nk^2\binom{2k}{k},\\ D_n&=\sum_{k=0}^n\binom{2k}{k}\binom{2n-2k}{n-k}\binom nk^2.\end{aligned}\tag{1.1}$$

Here $\{A_n\}$ are called Apéry numbers since Apéry [1] used it to prove $\zeta(3)$ is irrational in 1979, and $\{D_n\}$ are called Domb numbers. See [2,4,6,7,24]. Such sequences appear as coefficients in various series for $1/\pi$. For example,

$$\sum_{k=0}^{\infty}\frac{9k+2}{50^k}\binom{2k}{k}f_k=\frac{25}{2\pi} \quad\text{and}\quad \sum_{k=0}^{\infty}\frac{5k+1}{64^k}D_k=\frac{8}{\sqrt3\pi}.$$

Let $p>3$ be a prime, $m\in\mathbb Z_p$ and $m\not\equiv 0,-4,-8 \pmod p$. In this paper, we show that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{m}{(m+8)^2}\Big)^kf_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{m}{(m+4)^3}\Big)^k \pmod p.$$

Let $x\in\mathbb{Z}_p$, $x\not\equiv 0,-1,-\frac13\pmod p$ and $\left(\frac{9x^2+14x+9}{p}\right)=1$. We also show that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9x^2+14x+9}\Big)^k f_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{x}{9(1+3x)^4}\Big)^k \pmod p.$$

As consequences we determine $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{m^k}\pmod p$ for $m=5,-16,16,32,-49,50,96$ and $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{4^k}\pmod p$. As examples, for any prime $p>5$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k}\equiv 4x^2\pmod p\quad\text{for}\quad p=x^2+9y^2,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{16^k}\equiv 4x^2\pmod p\quad\text{for}\quad p=x^2+5y^2,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{5^k}\equiv 4x^2\pmod p\quad\text{for}\quad p=x^2+15y^2,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{32^k}\equiv 4x^2\pmod p\quad\text{for}\quad p=x^2+6y^2.$$

Let $p>3$ be a prime and $u\in\mathbb{Z}_p$ with $u\not\equiv 0\pmod p$. We show that

$$\begin{aligned}&-\Big(\frac p3\Big)\sum_{n=0}^{p-1}f_n\Big(-\frac{1+3u}{4}\Big)^n\\&\equiv\sum_{x=0}^{p-1}\Big(\frac{x^3-3u(u^3+6u+2)x-(2u^6-30u^4-10u^3-9u^2-6u-1)}{p}\Big)\pmod p.\end{aligned}$$

We also prove that $\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv 0\pmod{p^2}$ for any prime $p\equiv 5\pmod 6$, and pose many conjectures concerning the sum $\sum_{k=0}^{p-1}\binom{2k}{k}f_k/m^k\pmod{p^2}$, where m is an integer not divisible by p.

2. Main results

Lemma 2.1 ([18, Lemma 2.4]). *Let p be an odd prime, $u,c_0,c_1,\ldots,c_{p-1}\in\mathbb{Z}_p$ and $u\not\equiv 1\pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k c_k\equiv\sum_{n=0}^{p-1}u^n\sum_{k=0}^{n}\binom nk\binom{n+k}{k}c_k\pmod p.$$

Proof. Note that $p\mid\binom{2k}{k}$ for $\frac p2<k<p$, $\binom{-x}{k}=(-1)^k\binom{x+k-1}{k}$ and $\binom nk\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$. Using Fermat's little theorem we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k c_k$$

$$\equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k} c_k u^k (1-u)^{p-1-2k} = \sum_{k=0}^{(p-1)/2} \binom{2k}{k} c_k u^k \sum_{r=0}^{p-1-2k} \binom{p-1-2k}{r} (-u)^r$$
$$= \sum_{n=0}^{p-1} u^n \sum_{k=0}^{n} \binom{2k}{k} c_k (-1)^{n-k} \binom{p-1-2k}{n-k} = \sum_{n=0}^{p-1} u^n \sum_{k=0}^{n} \binom{2k}{k} c_k \binom{n+k-p}{n-k}$$
$$\equiv \sum_{n=0}^{p-1} u^n \sum_{k=0}^{n} \binom{2k}{k} c_k \binom{n+k}{n-k} = \sum_{n=0}^{p-1} u^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} c_k \pmod p.$$

Thus the lemma is proved. $\square$

Lemma 2.2. *Let $p>3$ be a prime and $c_0, c_1, \ldots, c_{p-1} \in \mathbb{Z}_p$. Then*

$$\sum_{n=0}^{p-1}\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} c_k \equiv \sum_{k=0}^{p-1} \frac{p}{2k+1}(-1)^k c_k \pmod{p^3}.$$

Proof. Since

$$\begin{aligned}\sum_{k=0}^{m} \binom{x}{k}(-1)^k &= \sum_{k=0}^{m} \binom{x-1}{k}(-1)^k + \sum_{k=1}^{m} \binom{x-1}{k-1}(-1)^k \\ &= \sum_{r=0}^{m} \binom{x-1}{r}(-1)^r - \sum_{r=0}^{m-1} \binom{x-1}{r}(-1)^r \\ &= \binom{x-1}{m}(-1)^m = \binom{m-x}{m}\end{aligned}$$

and $\binom{n}{k}\binom{n+k}{k} = \binom{2k}{k}\binom{n+k}{2k}$ we see that

$$\begin{aligned}&\sum_{n=0}^{p-1}\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} c_k \\ &= \sum_{k=0}^{p-1}\sum_{n=k}^{p-1} \binom{2k}{k}\binom{n+k}{2k} c_k = \sum_{k=0}^{p-1} \binom{2k}{k} c_k \sum_{r=0}^{p-1-k} \binom{2k+r}{2k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} c_k \sum_{r=0}^{p-1-k} \binom{-2k-1}{r}(-1)^r = \sum_{k=0}^{p-1} \binom{2k}{k} c_k \binom{p+k}{p-1-k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} c_k \binom{p+k}{2k+1} = \sum_{k=0}^{p-1} \frac{p}{2k+1} c_k \frac{(p^2-1^2)\cdots(p^2-k^2)}{k!^2}.\end{aligned}$$

Thus,

$$
\begin{aligned}
&\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k\\
&=\frac{(p^2-1^2)\cdots(p^2-(\frac{p-1}{2})^2)}{(\frac{p-1}{2})!^2}c_{\frac{p-1}{2}}+p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{c_k}{2k+1}\cdot\frac{(p^2-1^2)\cdots(p^2-k^2)}{k!^2}\\
&\equiv(-1)^{\frac{p-1}{2}}\Big(1-p^2\sum_{r=1}^{(p-1)/2}\frac{1}{r^2}\Big)c_{\frac{p-1}{2}}+p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^kc_k}{2k+1}\pmod{p^3}.
\end{aligned}
$$

By [13, Theorem 5.2], $\sum_{r=1}^{(p-1)/2}\frac{1}{r^2}\equiv 0\pmod p$. Thus the result follows. □

Example 2.1. Let $\{P_n(x)\}$ be the famous Legendre polynomials given by

$$P_0(x)=1,\ P_1(x)=x,\ (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)\ (n\ge 1).$$

It is well known that ([8, p.170])

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(\frac{x-1}{2}\Big)^k.$$

Thus, applying Lemma 2.2 we see that for any prime $p>3$ and $x\in\mathbb{Z}_p$,

$$\sum_{n=0}^{p-1}P_n(x)\equiv\sum_{k=0}^{p-1}\frac{p}{2k+1}\Big(\frac{1-x}{2}\Big)^k\pmod{p^3}. \tag{2.1}$$

Lemma 2.3 ([3, (2.19), p.1305 and (2.27)]. *Let n be a nonnegative integer. Then*

$$A_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}f_k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}a_k$$

and

$$\frac{D_n}{8^n}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{f_k}{(-8)^k}.$$

Lemma 2.3 can be verified straightforward by using Maple and the method in [5] to compare the recurrence relations for both sides.

Theorem 2.1. *Let p be an odd prime, $m\in\mathbb{Z}_p$ and $m\not\equiv 0,-4,-8\pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{m}{(m+8)^2}\Big)^kf_k\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}\Big(\frac{m}{(m+4)^3}\Big)^k\pmod p.$$

Proof. Taking $c_k = \frac{f_k}{(-8)^k}$ in Lemma 2.1 and then applying Lemma 2.3 we see that for $u \in \mathbb{Z}_p$ with $u \not\equiv 1 \pmod p$,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{u}{(1-u)^2}\Big)^k \frac{f_k}{(-8)^k} \equiv \sum_{n=0}^{p-1} u^n \frac{D_n}{8^n} \pmod p.$$

Now substituting u with $-\frac{8}{m}$ in the above formula we deduce that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{m}{(m+8)^2}\Big)^k f_k \equiv \sum_{n=0}^{p-1} \frac{D_n}{(-m)^n} \pmod p. \tag{2.2}$$

By [16, Theorem 3.1],

$$\sum_{n=0}^{p-1} \frac{D_n}{(-m)^n} \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{3k}{k} \Big(\frac{m}{(m+4)^3}\Big)^k \pmod p.$$

Thus the theorem is proved. □

Theorem 2.2. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{f_k}{50^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Proof. Taking $m = 2$ in Theorem 2.1 we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{f_k}{50^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \pmod p.$$

From [12] and [20] we know that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Thus the result follows. □

Theorem 2.3. *Let p be a prime with $p \equiv \pm 1 \pmod 8$. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{f_k}{32^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 \pmod p & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. Taking $m = 8$ in Theorem 2.1 we see that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} f_k}{32^k} \equiv \sum_{n=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{216^k} \pmod p.$$

Now applying [14, Theorem 4.5] we deduce the result. □

Theorem 2.4. *Let p be a prime with $p \equiv \pm 1 \pmod 5$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-49)^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1,19 \pmod{30} \text{ and so } p = x^2+15y^2, \\ 0 \pmod p & \text{if } p \equiv 11,29 \pmod{30}. \end{cases}$$

Proof. Taking $m = -1$ in Theorem 2.1 we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-49)^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k} \pmod p.$$

Now applying [14, Theorem 4.6] we deduce the result. □

Theorem 2.5. *Let p be a prime such that $p \equiv 1,19 \pmod{30}$ and so $p = x^2+15y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{5^k} \equiv 4x^2 \pmod p.$$

Proof. Note that $\left(\frac{-15}{p}\right) = \left(\frac{-3}{p}\right)\left(\frac{5}{p}\right) = \left(\frac{p}{3}\right)\left(\frac{p}{5}\right) = 1$. Let $t \in \{1,2,\ldots,\frac{p-1}{2}\}$ be given by $t^2 \equiv -15 \pmod p$ and $m = (-11+3t)/2$. Then $m \not\equiv 0 \pmod p$ and $\frac{64}{m} \equiv \frac{-11-3t}{2} \pmod p$ and so

$$\frac{(m+8)^2}{m} = 16 + m + \frac{64}{m} \equiv 16 + \frac{-11+3t}{2} + \frac{-11-3t}{2} = 5 \pmod p.$$

Also,

$$\frac{(m+4)^3}{m} \equiv \frac{(\frac{-3+3t}{2})^3}{\frac{-11+3t}{2}} \equiv -27 \pmod p.$$

Now applying Theorem 2.1 and [14, Theorem 4.6] we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{5^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k} \equiv 4x^2 \pmod p.$$

This proves the theorem. □

Remark 2.1 Let p be an odd prime. Taking $m = -16$ in Theorem 2.1 we deduce the congruence for $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-4)^k} \pmod p$.

Theorem 2.6. *Let p be an odd prime and $u \in \mathbb{Z}_p$.*
(i) *If $u \not\equiv 1 \pmod p$, then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k f_k \equiv \sum_{n=0}^{p-1} A_n u^n \pmod p.$$

(ii) *If* $u \not\equiv -1 \pmod p$, *then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1+u)^2}\Big)^k a_k \equiv \sum_{n=0}^{p-1} A_n u^n \pmod p.$$

Proof. Taking $c_k = f_k$ in Lemma 2.1 and then applying Lemma 2.3 we obtain (i). Taking $c_k = (-1)^k a_k$ in Lemma 2.1 and then applying Lemma 2.3 we see that for $u \not\equiv 1 \pmod p$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k (-1)^k a_k \equiv \sum_{n=0}^{p-1} u^n\cdot(-1)^n A_n \pmod p.$$

Now substituting u with $-u$ in the above we deduce (ii), which completes the proof. □

Theorem 2.7. *Let* $p>3$ *be a prime,* $x\in\mathbb{Z}_p$, $x\not\equiv 0,-1,-\frac13 \pmod p$ *and* $\big(\frac{9x^2+14x+9}{p}\big)=1$. *Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9x^2+14x+9}\Big)^k f_k \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{x}{9(1+3x)^4}\Big)^k \pmod p.$$

Proof. Let $v\in\{1,2,\ldots,\frac{p-1}{2}\}$ be given by $v^2\equiv 9x^2+14x+9 \pmod p$, and let

$$u=\frac{2x+v^2+3(x+1)v}{2x}.$$

Then $u\in\mathbb{Z}_p$. Since $v^2\equiv 9x^2+14x+9\not\equiv 9(x+1)^2 \pmod p$ we have $v\not\equiv \pm 3(x+1) \pmod p$. Thus $u\not\equiv 1 \pmod p$. If $u\equiv -1 \pmod p$, then $v^2+3(x+1)v\equiv -4x \pmod p$ and so $9(x+1)^2\equiv v^2+4x\equiv -3(x+1)v \pmod p$. As $x+1\not\equiv 0 \pmod p$ we have $v\equiv -3(x+1) \pmod p$ and get a contradiction. Thus $u\not\equiv -1 \pmod p$. Note that

$$\begin{aligned}&\frac{2x+v^2+3(x+1)v}{2x}\cdot\frac{2x+v^2-3(x+1)v}{2x}\\&=\frac{(2x+v^2)^2-9(x+1)^2v^2}{4x^2}\equiv\frac{(9x^2+16x+9)^2-9(x+1)^2(9x^2+14x+9)}{4x^2}\\&=\frac{(9x^2+16x+9)^2-(9x^2+16x+9+2x)(9x^2+16x+9-2x)}{4x^2}=1 \pmod p.\end{aligned}$$

We see that $u\not\equiv 0 \pmod p$ and

$$u+\frac1u\equiv\frac{2x+v^2+3(x+1)v}{2x}+\frac{2x+v^2-3(x+1)v}{2x}=\frac{2x+v^2}{x}\equiv\frac{9x^2+16x+9}{x} \pmod p.$$

Now, from the above and Theorem 2.6 we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9x^2+14x+9}\Big)^k f_k$$

$$\begin{aligned}
&\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(u+\frac{1}{u}-2)^k} = \sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1-u)^2}\Big)^k f_k \\
&\equiv \sum_{n=0}^{p-1} A_n u^n \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{u}{(1+u)^2}\Big)^k a_k \\
&\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{(u+\frac{1}{u}+2)^k} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9(x+1)^2}\Big)^k a_k \pmod p.
\end{aligned}$$

Taking $u=\frac{x}{9}$ in [16, Theorem 4.1] we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\Big(\frac{x}{9(x+1)^2}\Big)^k a_k \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{x}{9(1+3x)^4}\Big)^k \pmod p.$$

Thus the result follows. □

Theorem 2.8. *Let p be a prime of the form $12k+1$ and so $p=x^2+9y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k} \equiv 4x^2 \pmod p.$$

Proof. Taking $x=-3$ in Theorem 2.7 we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{(-16)^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-12288)^k} \pmod p.$$

Now applying [15, Theorem 5.3] we deduce the result. □

Theorem 2.9. *Let $p>5$ be a prime such that $p\equiv 1,5,19,23 \pmod{24}$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{96^k} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p\equiv 1,19 \pmod{24} \text{ and so } p=x^2+2y^2,\\ 0 \pmod p & \text{if } p\equiv 5,23 \pmod{24}.\end{cases}$$

Proof. Since $(\frac{6}{p})=1$, taking $x=9$ in Theorem 2.7 we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{96^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}} \pmod p.$$

Now applying [16, Theorem 5.6] we deduce the result. □

Theorem 2.10. *Let p be a prime such that $p\equiv 1,9 \pmod{20}$ and so $p=x^2+5y^2$. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{16^k} \equiv 4x^2 \pmod p.$$

Proof. Let $t \in \{1, 2, \dots, \frac{p-1}{2}\}$ be given by $t^2 \equiv -5 \pmod p$, and $x = \frac{1+4t}{9}$. Then $x \not\equiv 0, -1, -\frac{1}{3} \pmod p$, $\frac{1}{x} \equiv \frac{1-4t}{9} \pmod p$ and so $\frac{9x^2+14x+9}{x} = 14 + 9(x + \frac{1}{x}) \equiv 16 \pmod p$. Thus, $(\frac{9x^2+14x+9}{p}) = (\frac{16x}{p}) = (\frac{1+4t}{p}) = (\frac{-1-4t}{p}) = (\frac{(2-t)^2}{p}) = 1$. Also,

$$\frac{9(1+3x)^4}{x} \equiv (1-4t)\big(1+\frac{1+4t}{3}\big)^4 \equiv (1-4t)\cdot\frac{4^4}{3^4}\cdot(-4)(1+4t) \equiv -1024 \pmod p.$$

Now applying Theorem 2.7 and [15, Theorem 5.5] we deduce that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{16^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-1024)^k} \equiv 4x^2 \pmod p.$$

This proves the theorem. □

Lemma 2.4. *Let $p > 3$ be a prime and $z \in \mathbb{Z}_p$ with $z \not\equiv -\frac{1}{4} \pmod p$. Then*

$$\sum_{n=0}^{p-1} f_n z^n \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{z}{(1+4z)^3}\Big)^k \pmod p.$$

Proof. From [22, (2.3)] we know that

$$f_n = \sum_{k=0}^{n}\binom{2k}{k}\binom{3k}{k}\binom{n+2k}{3k}(-4)^{n-k}. \tag{2.3}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{p-1} f_n z^n &= \sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{2k}{k}\binom{3k}{k}\binom{n+2k}{3k}(-4)^{n-k}z^n \\
&= \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{n=k}^{p-1}\binom{n+2k}{3k}(-4z)^{n-k} \\
&= \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{r=0}^{p-1-k}\binom{3k+r}{3k}(-4z)^r \\
&= \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{r=0}^{p-1-k}\binom{-3k-1}{r}(4z)^r \\
&\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k\sum_{r=0}^{p-1-k}\binom{p-1-3k}{r}(4z)^r \\
&= \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}z^k(1+4z)^{p-1-3k} \\
&\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{z}{(1+4z)^3}\Big)^k \pmod p.
\end{aligned}$$

This proves the lemma. $\square$

Theorem 2.11. *Let $p > 3$ be a prime and $u \in \mathbb{Z}_p$ with $u \not\equiv 0 \pmod p$. Then*

$$-\Big(\frac{p}{3}\Big)\sum_{n=0}^{p-1} f_n\Big(-\frac{1+3u}{4}\Big)^n$$
$$\equiv \sum_{x=0}^{p-1}\Big(\frac{x^3-3u(u^3+6u+2)x-(2u^6-30u^4-10u^3-9u^2-6u-1)}{p}\Big) \pmod p.$$

Proof. As $\binom{2k}{k}\binom{3k}{k} = \frac{(3k)!}{k!^3}$ we see that $p \mid \binom{2k}{k}\binom{3k}{k}$ for $\frac{p}{3} < k < p$. Set $t_1 = \frac{1+3u}{2u^3}$. Putting $z = -(1+3u)/4$ in Lemma 2.4 and then applying [14, Corollary 3.1] we see that

$$\sum_{n=0}^{p-1} f_n\Big(-\frac{1+3u}{4}\Big)^n$$
$$\equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\Big(\frac{t_1}{54}\Big)^k$$
$$\equiv -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4(1-t_1)-5)x+2(2(1-t_1)^2-14(1-t_1)+11)}{p}\Big)$$
$$= -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(4t_1+1)x+4t_1^2+20t_1-2}{p}\Big)$$
$$= -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{(\frac{x}{u^2})^3-3(\frac{2(1+3u)}{u^3}+1)\frac{x}{u^2}+\frac{(1+3u)^2}{u^6}+\frac{10(1+3u)}{u^3}-2}{p}\Big)$$
$$= -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(2(1+3u)u+u^4)x+(1+3u)^2+10(1+3u)u^3-2u^6}{p}\Big) \pmod p.$$

This yields the result. $\square$

Corollary 2.1. *For any prime $p > 5$,*

$$\sum_{n=0}^{p-1} f_n \equiv -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-5115x+115414}{p}\Big) \pmod p.$$

Proof. Taking $u = -5/3$ in Theorem 2.11 we deduce that

$$\sum_{n=0}^{p-1} f_n \equiv -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{1705}{27}x+\frac{115414}{729}}{p}\Big) = -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{(\frac{x}{9})^3-\frac{1705}{27}\cdot\frac{x}{9}+\frac{115414}{729}}{p}\Big)$$
$$= -\Big(\frac{p}{3}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-5115x+115414}{p}\Big) \pmod p. \qquad \square$$

Theorem 2.12. *Let $p > 3$ be a prime and $z \in \mathbb{Z}_p$ with $z \not\equiv \frac{1}{2} \pmod p$. Then*

$$\sum_{n=0}^{p-1} f_n z^n \equiv \sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\Big(\frac{z^2}{(1-2z)^3}\Big)^k \pmod p.$$

Proof. From [22, (2.2)] we know that

$$f_n = \sum_{k=0}^{[n/2]} \binom{2k}{k}\binom{3k}{k}\binom{n+k}{3k} 2^{n-2k}. \tag{2.4}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{p-1} f_n z^n &= \sum_{n=0}^{p-1}\sum_{k=0}^{[n/2]} \binom{2k}{k}\binom{3k}{k}\binom{n+k}{3k} 2^{n-2k} z^n \\
&= \sum_{k=0}^{(p-1)/2} \binom{2k}{k}\binom{3k}{k}\Big(\frac{z}{2}\Big)^k \sum_{n=2k}^{p-1} \binom{n+k}{3k}(2z)^{n-k} \\
&= \sum_{k=0}^{(p-1)/2} \binom{2k}{k}\binom{3k}{k}\Big(\frac{z}{2}\Big)^k \sum_{r=0}^{p-1-2k} \binom{3k+r}{3k}(2z)^{k+r} \\
&= \sum_{k=0}^{(p-1)/2} \binom{2k}{k}\binom{3k}{k} z^{2k} \sum_{r=0}^{p-1-2k} \binom{-3k-1}{r}(-2z)^r \\
&\equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}\binom{3k}{k} z^{2k} \sum_{r=0}^{p-1-2k} \binom{p-1-3k}{r}(-2z)^r \\
&= \sum_{k=0}^{(p-1)/2} \binom{2k}{k}\binom{3k}{k} z^{2k}(1-2z)^{p-1-3k} \\
&\equiv \sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\Big(\frac{z^2}{(1-2z)^3}\Big)^k \pmod p.
\end{aligned}$$

This is the result. □

Corollary 2.2. *Let $p > 3$ be a prime and $z \in \mathbb{Z}_p$ with $z \not\equiv \frac{1}{2}, -\frac{1}{4} \pmod p$. Then*

$$\sum_{k=0}^{[p/3]} \binom{2k}{k}\binom{3k}{k}\Big(\frac{z}{(1+4z)^3}\Big)^k \equiv \sum_{k=0}^{[p/3]} \binom{2k}{k}\binom{3k}{k}\Big(\frac{z^2}{(1-2z)^3}\Big)^k \pmod p.$$

Proof. As $p \mid \binom{2k}{k}\binom{3k}{k}$ for $\frac{p}{3} < k < p$, combining Lemma 2.4 with Theorem 2.12 yields the result. □

Taking $c_k = f_k, (-1)^k a_k, \frac{f_k}{(-8)^k}$ in Lemma 2.2 and then applying Lemma 2.3 we see that for any prime $p > 3$,

$$\sum_{n=0}^{p-1} A_n \equiv \sum_{k=0}^{p-1} \frac{p}{2k+1}(-1)^k f_k \pmod{p^3}, \tag{2.5}$$

$$\sum_{n=0}^{p-1} (-1)^n A_n \equiv \sum_{k=0}^{p-1} \frac{p}{2k+1} a_k \pmod{p^3}, \tag{2.6}$$

$$\sum_{n=0}^{p-1} \frac{D_n}{8^n} \equiv \sum_{k=0}^{p-1} \frac{p}{2k+1} \cdot \frac{f_k}{8^k} \pmod{p^3}. \tag{2.7}$$

It is known that ([11, Lemma 2.6])

$$f_k \equiv (-8)^k f_{p-1-k} \pmod{p} \quad \text{for} \quad k = 0, 1, \ldots, p-1. \tag{2.8}$$

Thus, from (2.5) and (2.7) we see that

$$\begin{aligned}
\sum_{n=0}^{p-1} \frac{D_n}{8^n} &\equiv 8^{-\frac{p-1}{2}} f_{\frac{p-1}{2}} + p \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{1}{2k+1} \cdot \frac{f_k}{8^k} \\
&= 8^{-\frac{p-1}{2}} f_{\frac{p-1}{2}} + p \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{1}{2(p-1-k)+1} \cdot \frac{f_{p-1-k}}{8^{p-1-k}} \\
&\equiv 8^{-\frac{p-1}{2}} f_{\frac{p-1}{2}} + p \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{1}{-2k-1} \cdot \frac{f_k/(-8)^k}{8^{p-1-k}} \\
&\equiv 8^{-\frac{p-1}{2}} f_{\frac{p-1}{2}} - p \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{p-1} \frac{1}{2k+1}(-1)^k f_k \\
&= ((-1)^{\frac{p-1}{2}} + 8^{-\frac{p-1}{2}}) f_{\frac{p-1}{2}} - \sum_{k=0}^{p-1} \frac{p}{2k+1}(-1)^k f_k \\
&= \frac{1 + (-8)^{\frac{p-1}{2}}}{8^{\frac{p-1}{2}}} f_{\frac{p-1}{2}} - \sum_{n=0}^{p-1} A_n \pmod{p^2}.
\end{aligned}$$

If $p \equiv 5, 7 \pmod 8$, then $(-8)^{(p-1)/2} \equiv -1 \pmod p$ and so $p \mid f_{\frac{p-1}{2}}$ by (2.8). Hence $(1 + (-8)^{\frac{p-1}{2}}) f_{\frac{p-1}{2}} \equiv 0 \pmod{p^2}$ and so

$$\sum_{n=0}^{p-1} \frac{D_n}{8^n} \equiv -\sum_{n=0}^{p-1} A_n \pmod{p^2} \quad \text{for} \quad p \equiv 5, 7 \pmod 8. \tag{2.9}$$

Theorem 2.13. *Let p be a prime with $p \equiv 5 \pmod 6$. Then*

$$\sum_{n=0}^{p-1} \frac{D_n}{4^n} \equiv 0 \pmod{p^2}.$$

Proof. By [16, Lemma 3.1],

$$D_n = \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{3k}{k} \binom{n+k}{3k} 4^{n-2k}.$$

Thus, applying the identity in the proof of Lemma 2.2 we deduce that

$$\begin{aligned}
\sum_{n=0}^{p-1} \frac{D_n}{4^n} &= \sum_{n=0}^{p-1} \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{3k}{k} \binom{n+k}{3k} 4^{-2k} \\
&= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{16^k} \sum_{n=2k}^{p-1} \binom{n+k}{3k} \\
&= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{16^k} \sum_{r=0}^{p-1-2k} \binom{3k+r}{r} \\
&= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{16^k} \sum_{r=0}^{p-1-2k} \binom{-3k-1}{r} (-1)^r \\
&= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{16^k} \binom{p+k}{p-1-2k} = \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{16^k} \binom{p+k}{3k+1}.
\end{aligned}$$

For $1 \le k \le \frac{p-1}{2}$ we see that $p \nmid 3k+1$ and so

$$\begin{aligned}
\binom{3k}{k}\binom{p+k}{3k+1} &= \frac{p}{3k+1} \cdot \frac{(p^2-1^2)\cdots(p^2-k^2)(p-(k+1))\cdots(p-2k)}{k!(2k)!} \\
&\equiv \frac{p}{3k+1} \pmod{p^2}.
\end{aligned}$$

Hence

$$\sum_{n=0}^{p-1} \frac{D_n}{4^n} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \cdot \frac{p}{3k+1} \pmod{p^2}.$$

For $a \in \mathbb{Z}_p$ let

$$S_p(a) = \sum_{k=0}^{(p-1)/2} \binom{a}{k} \binom{-1-a}{k} \frac{1}{3k+1}.$$

By [17, (2.2)],

$$(3a+1)S_p(a) - (3a-1)S_p(a-1) = 2\binom{a-1}{\frac{p-1}{2}}\binom{-a-1}{\frac{p-1}{2}}.$$

For $a\not\equiv 0\pmod p$ we see that

$$\binom{a-1}{\frac{p-1}{2}}\binom{-a-1}{\frac{p-1}{2}}=\frac{(1^2-a^2)(2^2-a^2)\cdots((\frac{p-1}{2})^2-a^2)}{(\frac{p-1}{2}!)^2}\equiv 0\pmod p.$$

Since $p\equiv 2\pmod 3$, we have $p\nmid 3k+1$ for $1\le k\le \frac{p-1}{2}$. Hence $S_p(a)\in\mathbb{Z}_p$ and so

$$S_p(a)\equiv\frac{3a-1}{3a+1}S_p(a-1)=\frac{2-6a}{-2-6a}S_p(a-1)\pmod p\quad\text{for}\quad a\not\equiv 0,-\frac13\pmod p.$$

Therefore,

$$\begin{aligned}S_p\Big(-\frac12\Big)&\equiv\frac51 S_p\Big(-\frac12-1\Big)\equiv\frac51\cdot\frac{11}{7}S_p\Big(-\frac12-2\Big)\\&\equiv\cdots\equiv\frac{5\cdot 11\cdots p}{1\cdot 7\cdots(p-4)}S_p\Big(-\frac12-\frac{p+1}{6}\Big)\equiv 0\pmod p.\end{aligned}$$

Note that $\binom{-\frac12}{k}=\binom{2k}{k}/(-4)^k$. For $p\equiv 2\pmod 3$ we see that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k(3k+1)}=\sum_{k=0}^{(p-1)/2}\binom{-1/2}{k}^2\frac{1}{3k+1}=S_p\Big(-\frac12\Big)\equiv 0\pmod p$$

and so $\sum_{n=0}^{p-1}\frac{D_n}{4^n}\equiv 0\pmod {p^2}$. This proves the theorem. □

Remark 2.2 Theorem 2.13 was conjectured by Z.W. Sun in [21, Conjecture 5.2].

Theorem 2.14. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{4^k}\equiv\begin{cases}4x^2\pmod p & \text{if } p\equiv 1,3\pmod 8 \text{ and so } p=x^2+2y^2,\\ 0\pmod p & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Proof. Taking $u=1$ in Theorem 2.6(ii) we see that

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{4^k}\equiv\sum_{n=0}^{p-1}A_n\pmod p.$$

By [19, Corollary 1.2],

$$\sum_{n=0}^{p-1}A_n\equiv\begin{cases}4x^2\pmod p & \text{if } p\equiv 1,3\pmod 8 \text{ and so } p=x^2+2y^2,\\ 0\pmod p & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Thus the theorem is proved. □

Remark 2.3 Let p be an odd prime, and $m\in\mathbb{Z}_p$ with $m\not\equiv 0\pmod p$. For conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}\frac{a_k}{m^k}\pmod{p^2}$, see [21, Conjecture 7.8] and [16, Conjectures 6.4-6.6].

3. Conjectures involving Franel numbers

By calculations with Maple, we pose the following conjectures:

Conjecture 3.1. *Let p be an odd prime. If $p \equiv 1,3 \pmod 8$ and so $p = x^2 + 2y^2$, then*

$$f_{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}}\big((3\cdot 2^{p-1}+1)x^2 - 2p\big) \pmod{p^2}$$

and

$$f_{\frac{p^2-1}{2}} \equiv 4x^4(3\cdot 2^{p-1}+1) - 16px^2 \pmod{p^2}.$$

Conjecture 3.2. *Let p be an odd prime. If $p \equiv 5,7 \pmod 8$, then $f_{\frac{p^2-1}{2}} \equiv p^2 \pmod{p^3}$ and $f_{\frac{p^r-1}{2}} \equiv 0 \pmod{p^r}$ for $r = 1,2,3,\ldots$.*

For any nonzero integer m and prime p let

$$F_p(m) = \sum_{k=0}^{p-1}\binom{2k}{k}\frac{f_k}{m^k}.$$

Conjecture 3.3. *Let p be an odd prime. Then*

$$F_p(-16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2+y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid y,\\ 2p - 4x^2 \pmod{p^2} & \text{if } p = x^2+y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid x-3,\\ 4(\frac{xy}{3})xy \pmod{p^2} & \text{if } p = x^2+y^2 \equiv 5 \pmod{12},\\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Conjecture 3.4. *Let $p > 3$ be a prime. Then*

$$F_p(96) \equiv \begin{cases} (\frac{p}{3})(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2+2y^2 \equiv 1,3 \pmod 8,\\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Conjecture 3.5. *Let $p > 5$ be a prime. Then*

$$F_p(50) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2+3y^2 \equiv 1 \pmod 3,\\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Conjecture 3.6. *Let $p > 5$ be a prime. Then*

$$F_p(16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,9 \pmod{20} \text{ and so } p = x^2+5y^2,\\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3,7 \pmod{20} \text{ and so } 2p = x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } p \equiv 11,13,17,19 \pmod{20}. \end{cases}$$

Conjecture 3.7. *Let $p > 3$ be a prime. Then*

$$F_p(32) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,7 \pmod{24} \text{ and so } p = x^2+6y^2,\\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5,11 \pmod{24} \text{ and so } p = 2x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } p \equiv 13,17,19,23 \pmod{24}. \end{cases}$$

Conjecture 3.8. *Let $p>7$ be a prime. Then*

$$F_p(5)\equiv F_p(-49)\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+15y^2\equiv 1,19\pmod{30},\\ 2p-12x^2\pmod{p^2} & \text{if } p=3x^2+5y^2\equiv 17,23\pmod{30},\\ 0\pmod{p^2} & \text{if } p\equiv 7,11,13,29\pmod{30}.\end{cases}$$

Conjecture 3.9. *Let $b\in\{7,11,19,31,59\}$ and*

$$f(b)=\begin{cases}-112 & \text{if } b=7,\\ -400 & \text{if } b=11,\\ -2704 & \text{if } b=19,\\ -24304 & \text{if } b=31,\\ -1123600 & \text{if } b=59.\end{cases}$$

If p is a prime with $p\neq 2,3,b$ and $p\nmid f(b)$, then

$$F_p(f(b))\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+3by^2,\\ 2p-12x^2\pmod{p^2} & \text{if } p=3x^2+by^2,\\ 2x^2-2p\pmod{p^2} & \text{if } 2p=x^2+3by^2,\\ 2p-6x^2\pmod{p^2} & \text{if } 2p=3x^2+by^2,\\ 0\pmod{p^2} & \text{if } (\frac{-3b}{p})=-1.\end{cases}$$

Conjecture 3.10. *Let $b\in\{5,7,13,17\}$ and*

$$f(b)=\begin{cases}320 & \text{if } b=5,\\ 896 & \text{if } b=7,\\ 10400 & \text{if } b=13,\\ 39200 & \text{if } b=17.\end{cases}$$

If p is a prime with $p\neq 2,3,b$ and $p\nmid f(b)$, then

$$F_p(f(b))\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+6by^2,\\ 8x^2-2p\pmod{p^2} & \text{if } p=2x^2+3by^2,\\ 2p-12x^2\pmod{p^2} & \text{if } p=3x^2+2by^2,\\ 2p-24x^2\pmod{p^2} & \text{if } p=6x^2+by^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6b}{p})=-1.\end{cases}$$

Conjecture 3.11. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{9k+4}{5^k}\binom{2k}{k}f_k\equiv 4\Big(\frac{p}{5}\Big)p\pmod{p^2}\quad\text{for}\quad p>5,$$

$$\sum_{k=0}^{p-1}\frac{5k+2}{16^k}\binom{2k}{k}f_k\equiv 2p\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{9k+2}{50^k}\binom{2k}{k}f_k\equiv 2\Big(\frac{-1}{p}\Big)p \pmod{p^2}\quad \textit{for}\quad p\neq 5,$$
$$\sum_{k=0}^{p-1}\frac{5k+1}{96^k}\binom{2k}{k}f_k\equiv \Big(\frac{-2}{p}\Big)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{6k+1}{320^k}\binom{2k}{k}f_k\equiv \Big(\frac{p}{15}\Big)p \pmod{p^2}\quad \textit{for}\quad p\neq 5,$$
$$\sum_{k=0}^{p-1}\frac{90k+13}{896^k}\binom{2k}{k}f_k\equiv 13\Big(\frac{p}{7}\Big)p \pmod{p^2}\quad \textit{for}\quad p\neq 7,$$
$$\sum_{k=0}^{p-1}\frac{102k+11}{10400^k}\binom{2k}{k}f_k\equiv 11\Big(\frac{p}{39}\Big)p \pmod{p^2}\quad \textit{for}\quad p\neq 5,13.$$

Conjecture 3.12. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{15k+4}{(-49)^k}\binom{2k}{k}f_k\equiv 4\Big(\frac{p}{3}\Big)p \pmod{p^2}\quad \textit{for}\quad p\neq 5,7,$$
$$\sum_{k=0}^{p-1}\frac{9k+2}{(-112)^k}\binom{2k}{k}f_k\equiv 2\Big(\frac{p}{7}\Big)p \pmod{p^2}\quad \textit{for}\quad p\neq 7,$$
$$\sum_{k=0}^{p-1}\frac{99k+17}{(-400)^k}\binom{2k}{k}f_k\equiv 17\Big(\frac{-1}{p}\Big)p \pmod{p^2}\quad \textit{for}\quad p\neq 5,$$
$$\sum_{k=0}^{p-1}\frac{855k+109}{(-2704)^k}\binom{2k}{k}f_k\equiv 109p\Big(\frac{-1}{p}\Big) \pmod{p^2}\quad \textit{for}\quad p\neq 13,$$
$$\sum_{k=0}^{p-1}\frac{585k+58}{(-24304)^k}\binom{2k}{k}f_k\equiv 58p\Big(\frac{-31}{p}\Big) \pmod{p^2}\quad \textit{for}\quad p\neq 7,31.$$

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