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CONGRUENCES CONCERNING LUCAS SEQUENCES

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Let p be a prime greater than 3. In this paper, by using expansions and congruences for Lucas sequences and the theory of cubic residues and cubic congruences, we establish some congruences for $\sum_{k=0}^{[p/4]}\binom{4k}{2k}/m^k$ and $\sum_{k=0}^{[p/3]}\binom{3k}{k}/m^k$ modulo p, where $[x]$ is the greatest integer not exceeding x, and m is a rational p-adic integer with $m\not\equiv 0 \pmod p$.

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1. Introduction

Congruences involving binomial coefficients are interesting, and they are connected with Fermat quotients, Lucas sequences, Legendre polynomials and binary quadratic forms. In 2006 Adamchuk [1] conjectured that for any prime $p\equiv 1 \pmod 3$,

$$\sum_{k=1}^{\frac{2}{3}(p-1)}\binom{2k}{k}\equiv 0 \pmod{p^2}.$$

So far this conjecture is still open. In 2010 Z.W. Sun and Tauraso [16] proved that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\equiv\Big(\frac{-3}{p}\Big) \pmod{p^2},$$

where $(\frac{a}{p})$ is the Jacobi symbol. In 2013 the author [13] proved the following conjecture of Z.W. Sun:

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3\equiv\begin{cases}0\pmod{p^2} & \text{if } p\equiv 3,5,6\pmod 7,\\ 4x^2-2p\pmod{p^2} & \text{if } p=x^2+7y^2\equiv 1,2,4\pmod 7,\end{cases}$$

where p is an odd prime and x and y are integers.

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. For a prime p let $\mathbb{Z}_p$ denote the set of those rational numbers whose denominator is not divisible by p. For $a,b,c\in\mathbb{Z}$ and a prime p, if there are integers x and y such that $p=ax^2+bxy+cy^2$, we briefly write that $p=ax^2+bxy+cy^2$. Let $\{P_n(x)\}$ be the Legendre polynomials given by $P_0(x)=1$, $P_1(x)=x$ and $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$ $(n\ge 1)$. For any prime $p>3$ and $t\in\mathbb{Z}_p$, in [11] and [13] the author showed that

$$P_{[\frac p3]}(t)\equiv\sum_{k=0}^{[p/3]}\binom{2k}{k}\binom{3k}{k}\Big(\frac{1-t}{54}\Big)^k\pmod p,$$

$$P_{[\frac p4]}(t)\equiv\sum_{k=0}^{[p/4]}\binom{2k}{k}\binom{4k}{2k}\Big(\frac{1-t}{128}\Big)^k\pmod p,$$

where $[x]$ is the greatest integer not exceeding x. Recently the author [12-13] also established many congruences for $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\binom{3k}{k}/m^k$ and $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\binom{4k}{2k}/m^k\pmod{p^2}$, where $m\in\mathbb{Z}_p$ and $m\not\equiv 0\pmod p$. Such congruences are concerned with binary quadratic forms.

Let $p>5$ be a prime. In [18] Zhao, Pan and Sun obtained the congruence $\sum_{k=1}^{p-1}2^k\binom{3k}{k}\equiv\frac65((-1)^{(p-1)/2}-1)\pmod p$. In [15] Z.W. Sun investigated $\sum_{k=0}^{p-1}\binom{3k}{k}/m^k\pmod p$ for $m\not\equiv 0\pmod p$. He gave explicit congruences in the cases $m=6,7,8,9,13,-\frac14,\frac{27}{4},\frac83$.

Suppose that $p>3$ is a prime and $k\in\{0,1,\ldots,p-1\}$. It is easy to see that

$$\binom{2k}{k}\equiv 0\pmod p\quad\text{for}\quad \frac p2<k<p,$$

$$\binom{4k}{2k}\equiv 0\pmod p\quad\text{for}\quad \frac p4<k<\frac p2,$$

$$\binom{3k}{k}\equiv 0\pmod p\quad\text{for}\quad \frac p3<k<\frac p2.$$

Thus, for any $m\in\mathbb{Z}_p$ with $m\not\equiv 0\pmod p$,

$$\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}}{m^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{4k}{2k}}{m^k}\pmod p\quad\text{and}\quad\sum_{k=0}^{[p/3]}\frac{\binom{3k}{k}}{m^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{3k}{k}}{m^k}\pmod p.$$

Let p be a prime greater than 3 and $m\in\mathbb{Z}_p$ with $m\not\equiv 0\pmod p$. Inspired by the above work, in this paper we study congruences for $\sum_{k=0}^{[p/4]}\binom{4k}{2k}/m^k$ and $\sum_{k=0}^{[p/3]}\binom{3k}{k}/m^k$

modulo p. Such congruences are concerned with Lucas sequences, binary quadratic forms and the theory of cubic residues and cubic congruences. As examples, we have the following typical results:

(1.1) Let p be a prime such that $p\equiv \pm 1,\pm 4 \pmod{17}$. Then

$$\sum_{k=0}^{[p/4]}(-1)^k\binom{4k}{2k}\equiv 17^{[p/4]} \pmod p.$$

(1.2) Let p be a prime of the form $4k+1$ and so $p=c^2+d^2$ with $c,d\in\mathbb{Z}$ and $2\mid d$. Let $a\in\mathbb{Z}$ with $p\nmid(16a^2+1)$. Then

$$\sum_{k=0}^{(p-1)/4}\binom{4k}{2k}(-a^2)^k\equiv\begin{cases}\big(\frac{c-4ad}{16a^2+1}\big) \pmod p & \text{if } \big(\frac{16a^2+1}{p}\big)=1,\\ 0 \pmod p & \text{if } \big(\frac{16a^2+1}{p}\big)=-1.\end{cases}$$

(1.3) Let $p>3$ be a prime. If $(\frac{p}{23})=-1$, then $x\equiv\sum_{k=0}^{[p/3]}\binom{3k}{k} \pmod p$ is the unique solution of the congruence $23x^3+3x+1\equiv 0 \pmod p$. If $(\frac{p}{23})=1$, then

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\equiv\begin{cases}1 \pmod p & \text{if } p=x^2+xy+52y^2,\ 8x^2+7xy+8y^2,\\ (39x-10y)/(23y) \pmod p & \text{if } p=13x^2+xy+4y^2\ne 13,\\ -(87x+19y)/(23y) \pmod p & \text{if } p=29x^2+5xy+2y^2\ne 29.\end{cases}$$

2. Congruences for $\sum_{k=0}^{[p/4]}\binom{4k}{2k}/m^k \pmod p$

For any numbers P and Q, let $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ be the Lucas sequences given by

$$U_0(P,Q)=0,\ U_1(P,Q)=1,\ U_{n+1}(P,Q)=PU_n(P,Q)-QU_{n-1}(P,Q)\ (n\ge 1),$$
$$V_0(P,Q)=2,\ V_1(P,Q)=P,\ V_{n+1}(P,Q)=PV_n(P,Q)-QV_{n-1}(P,Q)\ (n\ge 1).$$

It is well known that (see [17])

$$U_n(P,Q)=\begin{cases}\frac{1}{\sqrt{P^2-4Q}}\Big\{\big(\frac{P+\sqrt{P^2-4Q}}{2}\big)^n-\big(\frac{P-\sqrt{P^2-4Q}}{2}\big)^n\Big\} & \text{if } P^2-4Q\ne 0,\\ n\big(\frac{P}{2}\big)^{n-1} & \text{if } P^2-4Q=0,\end{cases}$$

$$V_n(P,Q)=\Big(\frac{P+\sqrt{P^2-4Q}}{2}\Big)^n+\Big(\frac{P-\sqrt{P^2-4Q}}{2}\Big)^n.$$

In particular, we have

$$U_n(2,1)=n \quad\text{and}\quad U_n(a+b,ab)=\frac{a^n-b^n}{a-b}\quad\text{for}\quad a\ne b. \tag{2.1}$$

As usual, the sequences $F_n = U_n(1,-1)$ and $L_n = V_n(1,-1)$ are called the Fibonacci sequence and the Lucas sequence, respectively. It is easily seen that (see [3, Lemma 1.7] or [17, (4.2.20)-(4.2.21)])

(2.2) $2U_{n+1}(P,Q) = PU_n(P,Q)+V_n(P,Q), \quad 2QU_{n-1}(P,Q) = PU_n(P,Q)-V_n(P,Q).$

Lemma 2.1 ([17, (4.2.39)]). *For $n \in \mathbb{N}$ we have*

$$U_{2n+1}(P,Q) = \sum_{k=0}^{n} \binom{n+k}{n-k}(-Q)^{n-k}P^{2k}.$$

Lemma 2.2 ([13, Lemma 2.1]). *Let p be an odd prime and $k \in \{0,1,\dots,[\frac{p}{4}]\}$. Then*

$$\binom{[\frac{p}{4}]+k}{[\frac{p}{4}]-k} \equiv \binom{4k}{2k}\frac{1}{(-64)^k} \pmod p.$$

Lemma 2.3. *Let p be an odd prime and $k \in \{0,1,\dots,\frac{p-1}{2}\}$. Then*

$$\binom{(p-1)/2}{k} \equiv \frac{1}{(-4)^k}\binom{2k}{k} \pmod p.$$

Proof. It is clear that $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$.

Theorem 2.1. *Let p be an odd prime and $P,Q \in \mathbb{Z}_p$ with $PQ \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k}\Big(\frac{P^2}{64Q}\Big)^k \equiv (-Q)^{-[p/4]}U_{\frac{p+(\frac{-1}{p})}{2}}(P,Q) \pmod p,$$

$$\sum_{k=0}^{[p/4]} \binom{4k}{2k}\Big(\frac{Q}{4P^2}\Big)^k \equiv \Big(\frac{P}{p}\Big)U_{\frac{p+1}{2}}(P,Q) \pmod p.$$

Proof. Using Lemmas 2.1 and 2.2 we see that

$$\begin{aligned} U_{2[\frac{p}{4}]+1}(P,Q) &= \sum_{k=0}^{[p/4]} \binom{[\frac{p}{4}]+k}{[\frac{p}{4}]-k}(-Q)^{[\frac{p}{4}]-k}P^{2k} \\ &\equiv (-Q)^{[\frac{p}{4}]}\sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k}}{(-64)^k}\Big(\frac{P^2}{-Q}\Big)^k \pmod p. \end{aligned}$$

Note that $2[\frac{p}{4}]+1 = (p+(\frac{-1}{p}))/2$. We deduce the first result.

Using Lemma 2.3 we see that

$$\begin{aligned}
V_{\frac{p-1}{2}}(P,(P^2-4Q)/4) &= \Big(\frac{P+2\sqrt{Q}}{2}\Big)^{\frac{p-1}{2}} + \Big(\frac{P-2\sqrt{Q}}{2}\Big)^{\frac{p-1}{2}} \\
&= \frac{1}{2^{\frac{p-1}{2}}}\sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}P^{\frac{p-1}{2}-k}\big((2\sqrt{Q})^k+(-2\sqrt{Q})^k\big) \\
&= \frac{2}{2^{\frac{p-1}{2}}}\sum_{k=0}^{[p/4]}\binom{\frac{p-1}{2}}{2k}P^{\frac{p-1}{2}-2k}(2\sqrt{Q})^{2k} \\
&\equiv \frac{2}{2^{\frac{p-1}{2}}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}(-4)^{-2k}P^{\frac{p-1}{2}-2k}(4Q)^k \\
&\equiv 2\Big(\frac{2P}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{Q}{4P^2}\Big)^k \pmod p.
\end{aligned}$$

By appealing to [7, Lemma 3.1(ii)] we have

$$U_{\frac{p+1}{2}}(P,Q)\equiv\frac{1}{2}\Big(\frac{2}{p}\Big)V_{\frac{p-1}{2}}(P,(P^2-4Q)/4)\equiv\Big(\frac{P}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{Q}{4P^2}\Big)^k \pmod p.$$

This completes the proof.

Corollary 2.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{16^k}\equiv\frac{1}{2}\Big(\frac{2}{p}\Big) \pmod p.$$

Proof. Taking $P=2$ and $Q=1$ in Theorem 2.1 and then applying (2.1) we deduce the result.

Theorem 2.2. *Let p be an odd prime and $x\in\mathbb{Z}_p$ with $x\not\equiv 0,1 \pmod p$.*

(i) *If $p\equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{x}{16}\Big)^k\equiv x^{\frac{p-1}{4}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16x)^k} \pmod p.$$

(ii) *If $p\equiv 3 \pmod 4$, then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16x)^k}\equiv\Big(1-\frac{1}{x}\Big)^{\frac{p-3}{4}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(1-x))^k} \pmod p.$$

Proof. If $p\equiv 1 \pmod 4$, by Theorem 2.1 we have

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}\Big(\frac{x}{16}\Big)^k\equiv\Big(\frac{2}{p}\Big)U_{\frac{p+1}{2}}(2,x)\equiv(-1)^{\frac{p-1}{4}}\cdot(-x)^{\frac{p-1}{4}}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16x)^k} \pmod p.$$

If $p\equiv 3\pmod 4$, from [7, Lemma 3.1(i)] we know that $U_{\frac{p-1}{2}}(2,x)\equiv -(\frac{2}{p})U_{\frac{p-1}{2}}(2,1-x)\pmod p$. Now applying Theorem 2.1 and the fact $(\frac{2}{p})=-(-1)^{(p-3)/4}$ we deduce that

$$\begin{aligned}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16x)^k}&\equiv(-x)^{-[p/4]}U_{\frac{p-1}{2}}(2,x)\equiv x^{-[p/4]}U_{\frac{p-1}{2}}(2,1-x)\\&\equiv x^{-[\frac p4]}(x-1)^{[\frac p4]}\sum_{k=0}^{[p/4]}\binom{4k}{2k}\frac{1}{(16(1-x))^k}\pmod p.\end{aligned}$$

So the theorem is proved.

Theorem 2.3. *Let p be an odd prime and $P,Q\in\mathbb{Z}_p$ with $PQ(P^2-4Q)\not\equiv 0\pmod p$ and $(\frac{Q}{p})=1$.*

(i) *If $(\frac{4Q-P^2}{p})=-1$, then $\sum_{k=0}^{[p/4]}\binom{4k}{2k}(\frac{P^2}{64Q})^k\equiv 0\pmod p$.*

(ii) *If $(\frac{P^2-4Q}{p})=-1$, then $\sum_{k=0}^{[p/4]}\binom{4k}{2k}(\frac{Q}{4P^2})^k\equiv 0\pmod p$.*

Proof. Since $(\frac{Q}{p})=1$, it is well known ([2]) that $U_{(p-(\frac{P^2-4Q}{p}))/2}(P,Q)\equiv 0\pmod p$. This together with Theorem 2.1 yields the result.

As an example, taking $P=2$ and $Q=\frac12$ in Theorem 2.3(i) we see that

$$\sum_{k=0}^{(p-3)/4}\frac{1}{8^k}\binom{4k}{2k}\equiv 0\pmod p\quad\text{for any prime } p\equiv 7\pmod 8. \tag{2.3}$$

Theorem 2.4. *Let p be an odd prime. Then*

$$\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}}{(-16)^k}\equiv\begin{cases}(-1)^{\frac{p-1}{8}}2^{\frac{p-1}{4}}\pmod p & \text{if } p\equiv 1\pmod 8,\\(-1)^{\frac{p-3}{8}}2^{\frac{p-3}{4}}\pmod p & \text{if } p\equiv 3\pmod 8,\\0\pmod p & \text{if } p\equiv 5\pmod 8,\\(-1)^{\frac{p+1}{8}}2^{\frac{p-3}{4}}\pmod p & \text{if } p\equiv 7\pmod 8.\end{cases}$$

Proof. Taking $P=2$ and $Q=-1$ in Theorem 2.1 we obtain

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}(-16)^{-k}\equiv U_{(p+(\frac{-1}{p}))/2}(2,-1)\pmod p.$$

Now applying [4, Theorem 2.3] or [7, (1.7)-(1.8)] we deduce the result.

Theorem 2.5. *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}}{(-64)^k}\equiv\begin{cases}(-1)^{[\frac{p+5}{10}]}5^{[\frac p4]}\pmod p & \text{if } p\equiv 1,3,7,9\pmod{20},\\2(-1)^{[\frac{p+5}{10}]}5^{\frac{p-3}{4}}\pmod p & \text{if } p\equiv 11,19\pmod{20},\\0\pmod p & \text{if } p\equiv 13,17\pmod{20}\end{cases}$$

and

$$\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}}{(-4)^k}\equiv\begin{cases}(-1)^{[\frac{p+5}{10}]}5^{[\frac{p}{4}]}\pmod p & \text{if } p\equiv 1,9,11,19\pmod{20},\\ -2(-1)^{[\frac{p+5}{10}]}5^{\frac{p-3}{4}}\pmod p & \text{if } p\equiv 3,7\pmod{20},\\ 0\pmod p & \text{if } p\equiv 13,17\pmod{20}.\end{cases}$$

Proof. Taking $P=1$ and $Q=-1$ in Theorem 2.1 we obtain

$$\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}}{(-64)^k}\equiv F_{\frac{p+(\frac{-1}{p})}{2}}\pmod p\quad\text{and}\quad\sum_{k=0}^{[p/4]}\frac{\binom{4k}{2k}}{(-4)^k}\equiv F_{\frac{p+1}{2}}\pmod p.$$

Now applying [14, Corollaries 1-2 and Theorem 2] we deduce the result.

Theorem 2.6. *Let p be an odd prime with $p\neq 17$.*

(i) *If $p\equiv 1\pmod 4$, then*

$$\sum_{k=0}^{[p/4]}(-1)^k\binom{4k}{2k}\equiv\begin{cases}0\pmod p & \text{if } p\equiv\pm3,\pm5,\pm6,\pm7\pmod{17},\\ 17^{(p-1)/4}\pmod p & \text{if } p\equiv\pm1,\pm4\pmod{17},\\ -17^{(p-1)/4}\pmod p & \text{if } p\equiv\pm2,\pm8\pmod{17}.\end{cases}$$

(ii) *If $p\equiv 3\pmod 4$, then*

$$\sum_{k=0}^{[p/4]}(-1)^k\binom{4k}{2k}\equiv\begin{cases}17^{(p-3)/4}\pmod p & \text{if } p\equiv\pm1,\pm4\pmod{17},\\ -17^{(p-3)/4}\pmod p & \text{if } p\equiv\pm2,\pm8\pmod{17},\\ 4\cdot 17^{(p-3)/4}\pmod p & \text{if } p\equiv\pm3,\pm5\pmod{17},\\ -4\cdot 17^{(p-3)/4}\pmod p & \text{if } p\equiv\pm6,\pm7\pmod{17}.\end{cases}$$

Proof. Taking $P=8$ and $Q=-1$ in Theorem 2.1 we see that

$$\sum_{k=0}^{[p/4]}(-1)^k\binom{4k}{2k}\equiv U_{\frac{p+(\frac{-1}{p})}{2}}(8,-1)\pmod p.\tag{2.4}$$

By (2.2), $U_{\frac{p+1}{2}}(8,-1)=4U_{\frac{p-1}{2}}(8,-1)+\frac12 V_{\frac{p-1}{2}}(8,-1)$. From the above and [10, Corollary 4.5] we deduce the result.

Lemma 2.4 ([5, Lemma 3.4]). *Let p be an odd prime and $P,Q\in\mathbb{Z}_p$ with $Q(P^2-4Q)\not\equiv 0\pmod p$. If $\left(\frac{Q}{p}\right)=1$ and $c^2\equiv Q\pmod p$ for $c\in\mathbb{Z}_p$, then*

$$U_{\frac{p+1}{2}}(P,Q)\equiv\begin{cases}\left(\dfrac{P-2c}{p}\right)\pmod p & \text{if } \left(\frac{P^2-4Q}{p}\right)=1,\\ 0\pmod p & \text{if } \left(\frac{P^2-4Q}{p}\right)=-1\end{cases}$$

and

$$U_{\frac{p-1}{2}}(P,Q)\equiv\begin{cases}0\pmod p & \text{if } \left(\frac{P^2-4Q}{p}\right)=1,\\ \dfrac1c\left(\dfrac{P-2c}{p}\right)\pmod p & \text{if } \left(\frac{P^2-4Q}{p}\right)=-1.\end{cases}$$

Theorem 2.7. *Let p be an odd prime and $a\in\mathbb{Z}_p$ with $16a^2\not\equiv 1 \pmod p$. Then*

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}a^{2k}\equiv\begin{cases}0 \pmod p & \text{if } (\frac{1-16a^2}{p})=-1,\\ (\dfrac{1-4a}{p}) \pmod p & \text{if } (\frac{1-16a^2}{p})=1.\end{cases}$$

Proof. Putting $P=8a$ and $Q=1$ in Theorem 2.1 we deduce that

$$\sum_{k=0}^{[p/4]}\binom{4k}{2k}a^{2k}\equiv(-1)^{[p/4]}U_{\frac{p+(\frac{-1}{p})}{2}}(8a,1) \pmod p.$$

By Lemma 2.4,

$$(-1)^{[\frac p4]}U_{\frac{p+(\frac{-1}{p})}{2}}(8a,1)\equiv\begin{cases}(-1)^{[\frac p4]}(\dfrac{8a-2}{p})=(\dfrac{1-4a}{p}) \pmod p & \text{if } (\frac{16a^2-1}{p})=(\frac{-1}{p}),\\ 0 \pmod p & \text{if } (\frac{16a^2-1}{p})=-(\frac{-1}{p}).\end{cases}$$

Now combining all the above we obtain the result.

Theorem 2.8. *Let p be a prime of the form $4k+1$ and $p=c^2+d^2$ with $c,d\in\mathbb{Z}$ and $2\mid d$. Let $b,m\in\mathbb{Z}$ with $gcd(b,m)=1$ and $p\nmid m(b^2+4m^2)$. Then*

$$\Big(\frac mp\Big)\sum_{k=0}^{(p-1)/4}\binom{4k}{2k}\Big(-\frac{b^2}{64m^2}\Big)^k\equiv\Big(\frac bp\Big)\sum_{k=0}^{(p-1)/4}\binom{4k}{2k}\Big(-\frac{m^2}{4b^2}\Big)^k$$
$$\equiv\begin{cases}(\dfrac{bc+2md}{b^2+4m^2}) \pmod p & \text{if } 2\nmid b \text{ and } (\frac{b^2+4m^2}{p})=1,\\ (-1)^{\frac{(\frac b2c+md)^2-1}{8}+\frac d2}(\dfrac{\frac b2c+md}{((\frac b2)^2+m^2)/2}) \pmod p & \text{if } 4\mid b-2 \text{ and } (\frac{b^2+4m^2}{p})=1,\\ (\dfrac{mc-\frac b2d}{\frac{b^2}{4}+m^2}) \pmod p & \text{if } 4\mid b \text{ and } (\frac{b^2+4m^2}{p})=1,\\ 0 \pmod p & \text{if } (\frac{b^2+4m^2}{p})=-1.\end{cases}$$

In particular, for $b=8a$ and $m=1$ we have

$$\sum_{k=0}^{(p-1)/4}\binom{4k}{2k}(-a^2)^k\equiv\Big(\frac{2a}{p}\Big)\sum_{k=0}^{(p-1)/4}\binom{4k}{2k}\frac{1}{(-256a^2)^k}$$
$$\equiv\begin{cases}(\dfrac{c-4ad}{16a^2+1}) \pmod p & \text{if } (\frac{16a^2+1}{p})=1,\\ 0 \pmod p & \text{if } (\frac{16a^2+1}{p})=-1.\end{cases}$$

Proof. Putting $P=b$ and $Q=-m^2$ in Theorem 2.1 we see that

$$U_{\frac{p+1}{2}}(b,-m^2)\equiv m^{\frac{p-1}{2}}\sum_{k=0}^{\frac{p-1}{4}}\binom{4k}{2k}\Big(-\frac{b^2}{64m^2}\Big)^k\equiv\Big(\frac bp\Big)\sum_{k=0}^{\frac{p-1}{4}}\binom{4k}{2k}\Big(-\frac{m^2}{4b^2}\Big)^k \pmod p.$$

Now applying [10, Theorem 3.2] we deduce the result.

Theorem 2.9. *Let p be a prime of the form $4k+1$ and $a\in\mathbb{Z}$ with $p\nmid(1+16a^2)(1-16a^2)$. Let $p=c^2+d^2$ with $c,d\in\mathbb{Z}$ and $2\mid d$. Then*

$$2\sum_{k=0}^{[p/8]}\binom{8k}{4k}a^{4k}\equiv\begin{cases}(\frac{1-4a}{p})+(\frac{c-4ad}{16a^2+1})\pmod p & \text{if } (\frac{1-16a^2}{p})=(\frac{1+16a^2}{p})=1,\\ (\frac{1-4a}{p})\pmod p & \text{if } (\frac{1-16a^2}{p})=-(\frac{1+16a^2}{p})=1,\\ (\frac{c-4ad}{16a^2+1})\pmod p & \text{if } (\frac{1-16a^2}{p})=-(\frac{1+16a^2}{p})=-1,\\ 0\pmod p & \text{if } (\frac{1-16a^2}{p})=(\frac{1+16a^2}{p})=-1.\end{cases}$$

Proof. Since

$$2\sum_{k=0}^{[p/8]}\binom{8k}{4k}a^{4k}=\sum_{k=0}^{(p-1)/4}\binom{4k}{2k}a^{2k}+\sum_{k=0}^{(p-1)/4}\binom{4k}{2k}(-1)^ka^{2k},$$

from Theorems 2.7 and 2.8 we deduce the result.

3. Congruences for $\sum_{k=0}^{[p/3]}\binom{3k}{k}/m^k\pmod p$.

Lemma 3.1 ([11, p.1920]). *Let $p>3$ be a prime and $k\in\{1,2,\ldots,[\frac p3]\}$. Then*

$$\binom{[\frac p3]+k}{[\frac p3]-k}\equiv\binom{3k}{k}\frac{1}{(-27)^k}\pmod p.$$

Theorem 3.1. *Let $p>3$ be a prime and $a,b\in\mathbb{Z}_p$ with $ab\not\equiv 0\pmod p$. Then*

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{b^{2k}}{a^k}\equiv(-3a)^{-[\frac p3]}U_{2[\frac p3]+1}(9b,3a)\pmod p.$$

Proof. Using Lemmas 2.1 and 3.1 we see that for $P,Q\in\mathbb{Z}_p$ with $PQ\not\equiv 0\pmod p$,

$$\begin{aligned}U_{2[\frac p3]+1}(P,Q)&=\sum_{k=0}^{[p/3]}\binom{[\frac p3]+k}{[\frac p3]-k}(-Q)^{[\frac p3]-k}P^{2k}\\ &\equiv(-Q)^{[\frac p3]}\sum_{k=0}^{[p/3]}\binom{3k}{k}\Big(\frac{P^2}{27Q}\Big)^k\pmod p.\end{aligned}\tag{3.1}$$

Now taking $P=9b$ and $Q=3a$ in (3.1) we deduce the result.

Lemma 3.2. *For $n\in\mathbb{N}$ we have $U_n(1,1)=(-1)^{n-1}(\frac n3)$.*

Proof. Set $\omega=(-1+\sqrt{-3})/2$. By (2.1), $U_n(1,1)=\frac{(-\omega)^n-(-\omega^2)^n}{-\omega-(-\omega^2)}=(-1)^{n-1}(\frac n3)$.

Theorem 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{[p/3]}\frac{\binom{3k}{k}}{27^k}\equiv\begin{cases}1 \pmod p & \text{if } p\equiv\pm1 \pmod 9,\\ -1 \pmod p & \text{if } p\equiv\pm2 \pmod 9,\\ 0 \pmod p & \text{if } p\equiv\pm4 \pmod 9.\end{cases}$$

Proof. Taking $a=\frac{1}{3}$ and $b=\frac{1}{9}$ in Theorem 3.1 and then applying Lemma 3.2 we deduce that

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{1}{27^k}\equiv(-1)^{[\frac{p}{3}]}U_{2[\frac{p}{3}]+1}(1,1)=(-1)^{[\frac{p}{3}]}\Big(\frac{2[\frac{p}{3}]+1}{3}\Big) \pmod p.$$

This yields the result.

Remark 3.1. Let $p>3$ be a prime. By (3.1) and (2.1) we have

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\Big(\frac{4}{27}\Big)^k\equiv(-1)^{[\frac{p}{3}]}U_{2[\frac{p}{3}]+1}(2,1)=(-1)^{[\frac{p}{3}]}\Big(2\Big[\frac{p}{3}\Big]+1\Big)\equiv\frac{1}{3} \pmod p$$

and

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\Big(\frac{2}{27}\Big)^k\equiv(-2)^{-[\frac{p}{3}]}U_{2[\frac{p}{3}]+1}(2,2)=\Big(\frac{3}{p}\Big) \pmod p.$$

Lemma 3.3. *Let $p>3$ be a prime and $P,Q\in\mathbb{Z}_p$ with $PQ\not\equiv 0 \pmod p$. Then*

$$U_{2[\frac{p}{3}]+1}(P,Q)\equiv\begin{cases}-Q^{1-\frac{p-(\frac{p}{3})}{3}}U_{\frac{p-(\frac{p}{3})}{3}-1}(P,Q) \pmod p & \text{if } (\frac{P^2-4Q}{p})=1,\\ -Q^{-\frac{p-(\frac{p}{3})}{3}}U_{\frac{p-(\frac{p}{3})}{3}+1}(P,Q) \pmod p & \text{if } (\frac{P^2-4Q}{p})=-1.\end{cases}$$

Proof. Since $2[\frac{p}{3}]+1=p-\frac{p-(\frac{p}{3})}{3}$ and $\frac{P\pm\sqrt{P^2-4Q}}{2}=\frac{Q}{(P\mp\sqrt{P^2-4Q})/2}$, we see that

$$\begin{aligned}U_{2[\frac{p}{3}]+1}(P,Q)&=\frac{1}{\sqrt{P^2-4Q}}\Big\{\Big(\frac{P+\sqrt{P^2-4Q}}{2}\Big)^{2[\frac{p}{3}]+1}-\Big(\frac{P-\sqrt{P^2-4Q}}{2}\Big)^{2[\frac{p}{3}]+1}\Big\}\\&=\frac{1}{\sqrt{P^2-4Q}}\Big\{\Big(\frac{P+\sqrt{P^2-4Q}}{2}\Big)^{p}\Big(\frac{(P-\sqrt{P^2-4Q})/2}{Q}\Big)^{\frac{p-(\frac{p}{3})}{3}}\\&\quad-\Big(\frac{P-\sqrt{P^2-4Q}}{2}\Big)^{p}\Big(\frac{(P+\sqrt{P^2-4Q})/2}{Q}\Big)^{\frac{p-(\frac{p}{3})}{3}}\Big\}.\end{aligned}$$

Since

$$\begin{aligned}\Big(\frac{P\pm\sqrt{P^2-4Q}}{2}\Big)^p&\equiv\frac{P^p\pm(\sqrt{P^2-4Q})^p}{2^p}\equiv\frac{P\pm\sqrt{P^2-4Q}(P^2-4Q)^{\frac{p-1}{2}}}{2}\\&\equiv\frac{P\pm(\frac{P^2-4Q}{p})\sqrt{P^2-4Q}}{2} \pmod p,\end{aligned}$$

from the above we have

$$U_{2[\frac{p}{3}]+1}(P,Q) \equiv \frac{Q^{-(p-(\frac{p}{3}))/3}}{\sqrt{P^2-4Q}}\Big\{\frac{P+(\frac{P^2-4Q}{p})\sqrt{P^2-4Q}}{2}\Big(\frac{P-\sqrt{P^2-4Q}}{2}\Big)^{\frac{p-(\frac{p}{3})}{3}}$$
$$-\frac{P-(\frac{P^2-4Q}{p})\sqrt{P^2-4Q}}{2}\Big(\frac{P+\sqrt{P^2-4Q}}{2}\Big)^{\frac{p-(\frac{p}{3})}{3}}\Big\} \pmod p.$$

If $(\frac{P^2-4Q}{p})=-1$, from the above we deduce that

$$U_{2[\frac{p}{3}]+1}(P,Q) \equiv -Q^{-\frac{p-(\frac{p}{3})}{3}}U_{\frac{p-(\frac{p}{3})}{3}+1}(P,Q) \pmod p.$$

If $(\frac{P^2-4Q}{p})=1$, from the above and the fact $\frac{P\pm\sqrt{P^2-4Q}}{2}=\frac{Q}{(P\mp\sqrt{P^2-4Q})/2}$ we see that

$$U_{2[\frac{p}{3}]+1}(P,Q) \equiv -Q^{1-\frac{p-(\frac{p}{3})}{3}}U_{\frac{p-(\frac{p}{3})}{3}-1}(P,Q) \pmod p.$$

So the lemma is proved.

Theorem 3.3. *Let $p>3$ be a prime and $a,b\in\mathbb{Z}_p$ with $ab\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{b^{2k}}{a^k} \equiv \begin{cases} (-3a)^{[\frac{p}{3}]+1}U_{\frac{p-(\frac{p}{3})}{3}-1}(9b,3a) \pmod p & \text{if } (\frac{81b^2-12a}{p})=1,\\ -(-3a)^{[\frac{p}{3}]}U_{\frac{p-(\frac{p}{3})}{3}+1}(9b,3a) \pmod p & \text{if } (\frac{81b^2-12a}{p})=-1.\end{cases}$$

Proof. From Theorem 3.1 and Lemma 3.3 we see that

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{b^{2k}}{a^k} \equiv (-3a)^{-[\frac{p}{3}]}U_{2[\frac{p}{3}]+1}(9b,3a)$$
$$\equiv \begin{cases} -(-3a)^{-[\frac{p}{3}]}\cdot(3a)^{1-\frac{p-(\frac{p}{3})}{3}}U_{\frac{p-(\frac{p}{3})}{3}-1}(9b,3a) \pmod p & \text{if } \Big(\frac{81b^2-12a}{p}\Big)=1,\\ -(-3a)^{-[\frac{p}{3}]}\cdot(3a)^{-\frac{p-(\frac{p}{3})}{3}}U_{\frac{p-(\frac{p}{3})}{3}+1}(9b,3a) \pmod p & \text{if } \Big(\frac{81b^2-12a}{p}\Big)=-1.\end{cases}$$

To see the result we note that $2[\frac{p}{3}]=p-1-\frac{p-(\frac{p}{3})}{3}$ and so

$$(3a)^{-[\frac{p}{3}]-\frac{p-(\frac{p}{3})}{3}}=(3a)^{[\frac{p}{3}]-(p-1)}\equiv(3a)^{[\frac{p}{3}]} \pmod p.$$

Corollary 3.1. *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{[p/3]}\frac{\binom{3k}{k}}{(-27)^k} \equiv \begin{cases} F_{\frac{p-(\frac{p}{3})}{3}-1} \pmod p & \text{if } (\frac{p}{5})=1,\\ -F_{\frac{p-(\frac{p}{3})}{3}+1} \pmod p & \text{if } (\frac{p}{5})=-1.\end{cases}$$

Proof. Taking $a=-\frac{1}{3}$ and $b=\frac{1}{9}$ in Theorem 3.3 we obtain the result.

Theorem 3.4. *Let $p>5$ be a prime, and let $\varepsilon_p=1,-1$ or 0 according as $p\equiv\pm1\pmod 9$, $p\equiv\pm2\pmod 9$ or $p\equiv\pm4\pmod 9$.*

(i) *If $p\equiv 1,4\pmod{15}$ and so $p=x^2+15y^2$ with $x,y\in\mathbb{Z}$, then*

$$2\sum_{k=0}^{[p/6]}\frac{\binom{6k}{2k}}{27^{2k}}-\varepsilon_p\equiv\sum_{k=0}^{[p/3]}\frac{\binom{3k}{k}}{(-27)^k}\equiv\begin{cases}1\pmod p & \text{if } 3\mid y,\\ (x-5y)/(10y)\pmod p & \text{if } 3\mid y-x.\end{cases}$$

(ii) *If $p\equiv 2,8\pmod{15}$ and so $p=5x^2+3y^2$ with $x,y\in\mathbb{Z}$, then*

$$2\sum_{k=0}^{[p/6]}\frac{\binom{6k}{2k}}{27^{2k}}-\varepsilon_p\equiv\sum_{k=0}^{[p/3]}\frac{\binom{3k}{k}}{(-27)^k}\equiv\begin{cases}1\pmod p & \text{if } 3\mid y,\\ -(x+y)/(2y)\pmod p & \text{if } 3\mid y-x.\end{cases}$$

Proof. By Theorem 3.2,

$$2\sum_{k=0}^{[p/6]}\frac{\binom{6k}{2k}}{27^{2k}}=\sum_{k=0}^{[p/3]}\Big(\frac{\binom{3k}{k}}{27^k}+\frac{\binom{3k}{k}}{(-27)^k}\Big)\equiv\varepsilon_p+\sum_{k=0}^{[p/3]}\frac{\binom{3k}{k}}{(-27)^k}\pmod p.$$

If $p=x^2+15y^2\equiv 1,4\pmod{15}$, by [6, Theorem 6.2] we have

$$(3.2)\quad F_{\frac{p-1}{3}}\equiv\begin{cases}0\pmod p & \text{if } 3\mid y,\\ -x/(5y)\pmod p & \text{if } 3\mid y-x\end{cases}\quad\text{and } L_{\frac{p-1}{3}}\equiv\begin{cases}2\pmod p & \text{if } 3\mid y,\\ -1\pmod p & \text{if } 3\nmid y.\end{cases}$$

If $p=5x^2+3y^2\equiv 2,8\pmod{15}$, by [6, Theorem 6.2] we have

$$(3.3)\qquad F_{\frac{p+1}{3}}\equiv\begin{cases}0\pmod p & \text{if } 3\mid y,\\ x/y\pmod p & \text{if } 3\mid y-x\end{cases}\quad\text{and } L_{\frac{p+1}{3}}\equiv\begin{cases}-2\pmod p & \text{if } 3\mid y,\\ 1\pmod p & \text{if } 3\nmid y.\end{cases}$$

Note that $2F_{n\pm1}=L_n\pm F_n$. From Corollary 3.1 and the above we deduce the result.

Theorem 3.5. *Let p be an odd prime with $p\equiv 1,2,4,8\pmod{15}$.*

(i) *If $p\equiv 1,4\pmod{15}$ and so $p=x^2+15y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{1}{3^k}\equiv\begin{cases}1\pmod p & \text{if } 3\mid y,\\ -(3x+5y)/(10y)\pmod p & \text{if } 3\mid y-x.\end{cases}$$

(ii) *If $p\equiv 2,8\pmod{15}$ and so $p=5x^2+3y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{1}{3^k}\equiv\begin{cases}1\pmod p & \text{if } 3\mid y,\\ (3x-y)/(2y)\pmod p & \text{if } 3\mid y-x.\end{cases}$$

Proof. It is known that $U_n(3,1)=F_{2n}=F_nL_n$. Thus, putting $a=b=\frac13$ in Theorem 3.3 we see that

$$\sum_{k=0}^{[p/3]}\frac{\binom{3k}{k}}{3^k}\equiv\begin{cases}-U_{\frac{p-1}{3}-1}(3,1)=-F_{\frac{p-1}{3}-1}L_{\frac{p-1}{3}-1}\pmod p & \text{if } p\equiv 1\pmod 3,\\ U_{\frac{p+1}{3}+1}(3,1)=F_{\frac{p+1}{3}+1}L_{\frac{p+1}{3}+1}\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}$$

It is easily seen that

$$2F_{n\pm1}=L_n\pm F_n\quad\text{and}\quad 2L_{n\pm1}=5F_n\pm L_n.$$

Thus, if $p=x^2+15y^2\equiv 1,4\pmod{15}$, using (3.2) we see that

$$F_{\frac{p-1}{3}-1}L_{\frac{p-1}{3}-1}=\frac14\big(L_{\frac{p-1}{3}}-F_{\frac{p-1}{3}}\big)\big(5F_{\frac{p-1}{3}}-L_{\frac{p-1}{3}}\big)\equiv\begin{cases}-1\pmod p & \text{if } 3\mid y,\\ (3x+5y)/(10y)\pmod p & \text{if } 3\mid y-x.\end{cases}$$

If $p=5x^2+3y^2\equiv 2,8\pmod{15}$, using (3.3) we see that

$$F_{\frac{p+1}{3}+1}L_{\frac{p+1}{3}+1}=\frac14\big(L_{\frac{p+1}{3}}+F_{\frac{p+1}{3}}\big)\big(5F_{\frac{p+1}{3}}+L_{\frac{p+1}{3}}\big)\equiv\begin{cases}1\pmod p & \text{if } 3\mid y,\\ (3x-y)/(2y)\pmod p & \text{if } 3\mid y-x.\end{cases}$$

Now combining all the above we obtain the result.

Theorem 3.6. *Let p be an odd prime with $(\frac{p}{13})=(\frac{p}{3})$. Then*

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{1}{(-3)^k}$$
$$\equiv\begin{cases}1\pmod p & \text{if } p=x^2+xy+88y^2,\ 10x^2+7xy+10y^2,\\ & \quad\text{or } 11x^2+xy+8y^2,\\ -(25x+10y)/(13y)\pmod p & \text{if } p=25x^2+7xy+4y^2,\\ (43x+12y)/(13y)\pmod p & \text{if } p=43x^2+37xy+10y^2\neq 43,\\ -(5x+8y)/(13y)\pmod p & \text{if } p=5x^2+3xy+18y^2\neq 5,\\ -(47x+9y)/(13y)\pmod p & \text{if } p=47x^2+5xy+2y^2\neq 47.\end{cases}$$

Proof. Taking $b=\frac13$ and $a=-\frac13$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\frac{1}{(-3)^k}$$
$$\equiv\begin{cases}U_{\frac{p-1}{3}-1}(3,-1)=-\frac12(3U_{\frac{p-1}{3}}(3,-1)-V_{\frac{p-1}{3}}(3,-1))\pmod p & \text{if } (\frac{13}{p})=1,\\ -U_{\frac{p+1}{3}+1}(3,-1)=-\frac12(3U_{\frac{p+1}{3}}(3,-1)+V_{\frac{p+1}{3}}(3,-1))\pmod p & \text{if } (\frac{13}{p})=-1.\end{cases}$$

Now applying [9, Corollary 6.7] we deduce the result.

Theorem 3.7. *Let p be an odd prime with $(\frac{p}{3})(\frac{p}{5})(\frac{p}{17})=1$. Then*

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}(-3)^k$$
$$\equiv\begin{cases}1\pmod p & \text{if } p=x^2+xy+64y^2,\ 3x^2+3xy+22y^2,\\ & \quad 8x^2+xy+8y^2 \text{ or } 5x^2+5xy+14y^2,\\ -(171x+74y)/(85y)\pmod p & \text{if } p=19x^2+7xy+4y^2\neq 19,\\ -(63x+65y)/(85y)\pmod p & \text{if } p=7x^2+5xy+10y^2\neq 7,\\ -(63x+13y)/(17y)\pmod p & \text{if } p=35x^2+5xy+2y^2,\\ (99x-29y)/(85y)\pmod p & \text{if } p=11x^2+3xy+6y^2\neq 11.\end{cases}$$

Proof. Taking $b=1$ and $a=-\frac{1}{3}$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}(-3)^k$$
$$\equiv\begin{cases} U_{\frac{p-1}{3}-1}(9,-1)=\dfrac{1}{2}(V_{\frac{p-1}{3}}(9,-1)-9U_{\frac{p-1}{3}}(9,-1)) \pmod p & \text{if } (\frac{85}{p})=1,\\ -U_{\frac{p+1}{3}+1}(9,-1)=-\dfrac{1}{2}(V_{\frac{p+1}{3}}(9,-1)+9U_{\frac{p+1}{3}}(9,-1)) \pmod p & \text{if } (\frac{85}{p})=-1.\end{cases}$$

Now applying [9, Corollary 6.9] we deduce the result.

Let (u,v) be the greatest common divisor of integers u and v. For $a,b,c\in\mathbb{Z}$ we use $[a,b,c]$ to denote the equivalence class containing the form $ax^2+bxy+cy^2$. It is well known that

$$(3.4)\qquad [a,b,c]=[c,-b,a]=[a,2ak+b,ak^2+bk+c]\quad\text{for}\quad k\in\mathbb{Z}.$$

We also use $H(d)$ to denote the form class group of discriminant d. Let $\omega=(-1+\sqrt{-3})/2$. Following [6] and [9] we use $(\frac{a+b\omega}{m})_3$ $(3\nmid m)$ to denote the cubic Jacobi symbol. For a prime $p>3$ and $k\in\mathbb{Z}_p$ with $k^2+3\not\equiv 0 \pmod p$, using [6, Corollary 6.1] we can easily determine $(\frac{k+1+2\omega}{p})_3$. In particular, by [6, Proposition 2.1] we have $(\frac{1+2\omega}{p})_3=1$.

For later convenience, following [9] we introduce the following notation.

Definition 3.1. *Suppose $u,v,d\in\mathbb{Z}$, $dv(u^2-dv^2)\neq 0$ and $(u,v)=1$. Let $u^2-dv^2=2^\alpha 3^r W (2\nmid W, 3\nmid W)$ and let w be the product of all distinct prime divisors of W. Define*

$$k_2(u,v,d)=\begin{cases}2 & \text{if } d\equiv 2,3 \pmod 4,\\ 2 & \text{if } d\equiv 1 \pmod 8,\ \alpha>0 \text{ and } \alpha\equiv 0,1 \pmod 3,\\ 1 & \text{otherwise},\end{cases}$$

$$k_3(u,v,d)=\begin{cases}3^{\mathrm{ord}_3 v+1} & \text{if } 3\mid r \text{ and } 3\nmid u,\\ 9 & \text{if } 3\nmid r \text{ and } 3\nmid u,\\ 3 & \text{if } 3\nmid r-2,\ 3\mid u \text{ and } 9\nmid u,\\ 1 & \text{otherwise}\end{cases}$$

and $k(u,v,d)=k_2(u,v,d)k_3(u,v,d)w/(u,w)$.

Lemma 3.4 ([9, Theorem 6.1 and Remark 6.1]). *Let $p>3$ be a prime, and $P,Q\in\mathbb{Z}$ with $p\nmid Q$ and $\big(\frac{-3(P^2-4Q)}{p}\big)=1$. Assume $P^2-4Q=df^2$ $(d,f\in\mathbb{Z})$ and $p=ax^2+bxy+cy^2$ with $a,b,c,x,y\in\mathbb{Z}$, $(a,6p\cdot 4Q/(P,f)^2)=1$ and $b^2-4ac=-3k^2d$, where $k=k(P/(P,f),f/(P,f),d)$. Then*

$$U_{(p-(\frac{p}{3}))/3}(P,Q)\equiv\begin{cases}0 \pmod p & \text{if } \Big(\frac{\frac{bf}{(P,f)}-\frac{kP}{(P,f)}(1+2\omega)}{a}\Big)_3=1,\\ -\dfrac{2ax+by}{kdfy}\big(\dfrac{-Q}{p}\big)(-Q)^{\frac{p-(\frac{p}{3})}{6}} \pmod p & \text{if } \Big(\frac{\frac{bf}{(P,f)}-\frac{kP}{(P,f)}(1+2\omega)}{a}\Big)_3=\omega,\\ \dfrac{2ax+by}{kdfy}\big(\dfrac{-Q}{p}\big)(-Q)^{\frac{p-(\frac{p}{3})}{6}} \pmod p & \text{if } \Big(\frac{\frac{bf}{(P,f)}-\frac{kP}{(P,f)}(1+2\omega)}{a}\Big)_3=\omega^2\end{cases}$$

and

$$V_{(p-(\frac p3))/3}(P,Q)\equiv\begin{cases}2\big(\frac p3\big)\big(\frac{-Q}{p}\big)(-Q)^{\frac{p-(\frac p3)}{6}}\pmod p & \text{if } \Big(\frac{\frac{bf}{(P,f)}-\frac{kP}{(P,f)}(1+2\omega)}{a}\Big)_3=1,\\ -\big(\frac p3\big)\big(\frac{-Q}{p}\big)(-Q)^{\frac{p-(\frac p3)}{6}}\pmod p & \text{if } \Big(\frac{\frac{bf}{(P,f)}-\frac{kP}{(P,f)}(1+2\omega)}{a}\Big)_3\ne1.\end{cases}$$

Moreover, the criteria for $p\mid U_{(p-(\frac p3))/3}(P,Q)$ and $V_{(p-(\frac p3))/3}(P,Q)\pmod p$ are also true when $p=a$.

Theorem 3.8. *Let $p>3$ be a prime with $(\frac{p}{23})=1$. Then*

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\equiv\begin{cases}1\pmod p & \text{if } p=x^2+xy+52y^2,\ 8x^2+7xy+8y^2,\\ (39x-10y)/(23y)\pmod p & \text{if } p=13x^2+xy+4y^2\ne13,\\ -(87x+19y)/(23y)\pmod p & \text{if } p=29x^2+5xy+2y^2\ne29.\end{cases}$$

Proof. Putting $a=b=1$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}\equiv\begin{cases}(-3)^{\frac{p-1}{3}+1}\cdot\frac16(9U_{\frac{p-1}{3}}(9,3)-V_{\frac{p-1}{3}}(9,3))\pmod p & \text{if } 3\mid p-1,\\ -(-3)^{\frac{p-2}{3}}\cdot\frac12(9U_{\frac{p+1}{3}}(9,3)+V_{\frac{p+1}{3}}(9,3))\pmod p & \text{if } 3\mid p-2.\end{cases}$$

Since $(\frac{p}{23})=1$ we have $(\frac{-3\cdot69}{p})=(\frac{-23}{p})=(\frac{p}{23})=1$ and $(\frac{69}{p})=(\frac p3)$. Thus p is represented by some class in $H(-207)$. From the theory of reduced forms we know that

$$H(-207)=\{[1,1,52],[8,7,8],[4,1,13],[4,-1,13],[2,1,26],[2,-1,26]\}.$$

Using (3.4) one can easily see that $[2,-1,26]=[2,-5,29]=[29,5,2]$ and $[8,7,8]=[8,23,23]=[23,-23,8]$. Note that

$$\Big(\frac{1-9(1+2\omega)}{1}\Big)_3=1,\quad \Big(\frac{-23-9(1+2\omega)}{23}\Big)_3=\Big(\frac{1+2\omega}{23}\Big)_3=1,$$
$$\Big(\frac{1-9(1+2\omega)}{13}\Big)_3=\Big(\frac{-3+1+2\omega}{13}\Big)_3=\omega,$$
$$\Big(\frac{5-9(1+2\omega)}{29}\Big)_3=\Big(\frac{-7+1+2\omega}{29}\Big)_3=\omega^2.$$

Since $k(9,1,69)=1$ by Definition 3.1, putting $P=9$, $Q=3$, $d=69$, $f=1$ and $k=1$ in Lemma 3.4 and applying the above we see that

$$U_{\frac{p-(\frac p3)}{3}}(9,3)\equiv\begin{cases}0\pmod p & \text{if } p=x^2+xy+52y^2,\ 8x^2+7xy+8y^2,\\ -\frac{26x+y}{69y}(-3)^{\frac{p-1}{6}}\pmod p & \text{if } p=13x^2+xy+4y^2\ne13,\\ -\frac{58x+5y}{69y}(-3)^{\frac{p+1}{6}}\pmod p & \text{if } p=29x^2+5xy+2y^2\ne29\end{cases}$$

and

$$V_{\frac{p-(\frac p3)}{3}}(9,3)\equiv\begin{cases}2(-3)^{(p-1)/6}\pmod p & \text{if } p=x^2+xy+52y^2\\ 2(-3)^{(p+1)/6}\pmod p & \text{if } p=8x^2+7xy+8y^2,\\ -(-3)^{(p-1)/6}\pmod p & \text{if } p=13x^2+xy+4y^2,\\ -(-3)^{(p+1)/6}\pmod p & \text{if } p=29x^2+5xy+2y^2.\end{cases}$$

Now combining all the above with the fact $(-3)^{(p-1)/2} \equiv (\frac{-3}{p}) = (\frac{p}{3}) \pmod p$ we deduce the result.

Theorem 3.9. *Let $p > 3$ be a prime with $(\frac{p}{31}) = 1$. Then*

$$\sum_{k=0}^{[p/3]} \binom{3k}{k}(-1)^k \equiv \begin{cases} 1 \pmod p & \text{if } p = x^2+xy+70y^2, 9x^2+9xy+10y^2 \\ & \quad \text{or } 8x^2+3xy+9y^2, \\ (15x-14y)/(31y) \pmod p & \text{if } p = 5x^2+xy+14y^2 \neq 5, \\ (21x-14y)/(31y) \pmod p & \text{if } p = 7x^2+xy+10y^2 \neq 7, \\ (57x-8y)/(31y) \pmod p & \text{if } p = 19x^2+5xy+4y^2 \neq 19, \\ -(105x+17y)/(31y) \pmod p & \text{if } p = 35x^2+xy+2y^2. \end{cases}$$

Proof. Putting $a = -1$ and $b = 1$ in Theorem 3.3 and applying (2.2) we see that

$$\sum_{k=0}^{[p/3]} \binom{3k}{k}(-1)^k \equiv \begin{cases} 3^{\frac{p-1}{3}+1} \cdot \frac{1}{(-6)}(9U_{\frac{p-1}{3}}(9,-3) - V_{\frac{p-1}{3}}(9,-3)) \pmod p & \text{if } 3 \mid p-1, \\ -3^{\frac{p-2}{3}} \cdot \frac{1}{2}(9U_{\frac{p+1}{3}}(9,-3) + V_{\frac{p+1}{3}}(9,-3)) \pmod p & \text{if } 3 \mid p-2. \end{cases}$$

Since $(\frac{p}{31}) = 1$ we have $(\frac{-3\cdot 93}{p}) = (\frac{-31}{p}) = (\frac{p}{31}) = 1$ and $(\frac{93}{p}) = (\frac{p}{3})$. Thus p is represented by some class in $H(-279)$. From the theory of reduced forms we know that

$$H(-279) = \{[1,1,70],[9,9,10],[2,1,35],[2,-1,35],[5,1,14],[5,-1,14], [7,1,10],[7,-1,10],[4,3,18],[4,-3,18],[8,3,9],[8,-3,9]\}.$$

Using (3.4) one can easily see that $[2,-1,35] = [35,1,2]$, $[4,3,18] = [4,-5,19] = [19,5,4]$, $[8,3,9] = [8,-29,35] = [35,29,8]$ and $[9,9,10] = [10,-9,9] = [10,31,31] = [31,-31,10]$. By [6, Example 2.1],

$$\Big(\frac{1-9(1+2\omega)}{1}\Big)_3 = 1, \quad \Big(\frac{-31-9(1+2\omega)}{31}\Big)_3 = \Big(\frac{1+2\omega}{31}\Big)_3 = 1,$$
$$\Big(\frac{29-9(1+2\omega)}{35}\Big)_3 = \Big(\frac{24+1+2\omega}{5}\Big)_3\Big(\frac{24+1+2\omega}{7}\Big)_3 = \omega^2\cdot\omega = 1,$$
$$\Big(\frac{1-9(1+2\omega)}{5}\Big)_3 = \Big(\frac{1+1+2\omega}{5}\Big)_3 = \omega, \quad \Big(\frac{1-9(1+2\omega)}{7}\Big)_3 = \Big(\frac{3+1+2\omega}{7}\Big)_3 = \omega,$$
$$\Big(\frac{5-9(1+2\omega)}{19}\Big)_3 = \Big(\frac{-9+1+2\omega}{19}\Big)_3 = \omega,$$
$$\Big(\frac{1-9(1+2\omega)}{35}\Big)_3 = \Big(\frac{-4+1+2\omega}{5}\Big)_3\Big(\frac{-4+1+2\omega}{7}\Big)_3 = \omega\cdot\omega = \omega^2.$$

Since $k(9,1,93) = 1$ by Definition 3.1, putting $P = 9$, $Q = -3$, $d = 93$, $f = 1$ and

$k=1$ in Lemma 3.4 and applying the above we see that

$$U_{\frac{p-(\frac{p}{3})}{3}}(9,-3)\equiv\begin{cases}0\pmod p & \text{if } p=x^2+xy+70y^2, 9x^2+9xy+10y^2\\ & \quad\text{or } 8x^2+3xy+9y^2,\\ -\frac{10x+y}{93y}(\frac{3}{p})3^{\frac{p+1}{6}}\pmod p & \text{if } p=5x^2+xy+14y^2\neq 5,\\ -\frac{14x+y}{93y}(\frac{3}{p})3^{\frac{p-1}{6}}\pmod p & \text{if } p=7x^2+xy+10y^2\neq 7,\\ -\frac{38x+5y}{93y}(\frac{3}{p})3^{\frac{p-1}{6}}\pmod p & \text{if } p=19x^2+5xy+4y^2\neq 19,\\ \frac{70x+y}{93y}(\frac{3}{p})3^{\frac{p+1}{6}}\pmod p & \text{if } p=35x^2+xy+2y^2\end{cases}$$

and

$$V_{\frac{p-(\frac{p}{3})}{3}}(9,-3)\equiv\begin{cases}2(\frac{3}{p})3^{(p-1)/6}\pmod p & \text{if } p=x^2+xy+70y^2, 9x^2+9xy+10y^2,\\ -2(\frac{3}{p})3^{(p+1)/6}\pmod p & \text{if } p=8x^2+3xy+9y^2,\\ (\frac{3}{p})3^{(p+1)/6}\pmod p & \text{if } p=5x^2+xy+14y^2, 35x^2+xy+2y^2,\\ -(\frac{3}{p})3^{(p-1)/6}\pmod p & \text{if } p=7x^2+xy+10y^2, 19x^2+5xy+4y^2.\end{cases}$$

Now combining all the above we deduce the result.

Theorem 3.10. *Let $p>3$ be a prime and $a\in\mathbb{Z}_p$ with $\big(\frac{a(4-27a)}{p}\big)=-1$. Then $x\equiv\sum_{k=0}^{[p/3]}\binom{3k}{k}a^k\pmod p$ is the unique solution of the cubic congruence $(27a-4)x^3+3x+1\equiv 0\pmod p$.*

Proof. As $(\frac{a(4-27a)}{p})=-1$ we have $(\frac{81a^2-12a}{p})=(\frac{-3}{p})(\frac{a(4-27a)}{p})=-(\frac{-3}{p})=-(\frac{p}{3})$. Thus putting $b=a$ in Theorem 3.3 we obtain

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}a^k\equiv\begin{cases}-(3a)^{\frac{p-1}{3}}U_{\frac{p-1}{3}+1}(9a,3a)\pmod p & \text{if } p\equiv 1\pmod 3,\\ (-3a)^{\frac{p-2}{3}+1}U_{\frac{p+1}{3}-1}(9a,3a)\pmod p & \text{if } p\equiv 2\pmod 3.\end{cases}$$

From [8, Theorem 2.1] or [9, Remark 6.1] we know that

$$U_{\frac{p-(\frac{p}{3})}{3}}(9a,3a)\equiv\frac{1}{27a-4}\Big(\frac{-a}{p}\Big)(3a)^{\frac{p-(\frac{p}{3})}{6}-1}(-3x_0^2+2x_0+18a)\pmod p$$

and

$$V_{\frac{p-(\frac{p}{3})}{3}}(9a,3a)\equiv\Big(\frac{3a}{p}\Big)(3a)^{\frac{p-(\frac{p}{3})}{6}-1}(x_0^2-6a)\pmod p,$$

where x_0 is the unique solution of the congruence $X^3-9aX-27a^2\equiv 0\pmod p$. Hence

$$9aU_{\frac{p-(\frac{p}{3})}{3}}(9a,3a)+\Big(\frac{p}{3}\Big)V_{\frac{p-(\frac{p}{3})}{3}}(9a,3a)$$

$$\equiv \frac{1}{27a-4}\Big(\frac{-a}{p}\Big)(3a)^{\frac{p-(\frac p3)}{6}-1}\big(9a(-3x_0^2+2x_0+18a)+(27a-4)(x_0^2-6a)\big)$$
$$= -\frac{2}{27a-4}\Big(\frac{-a}{p}\Big)(3a)^{\frac{p-(\frac p3)}{6}-1}(2x_0^2-9ax_0-12a) \pmod p.$$

Now putting $b=a$ in Theorem 3.3 and applying (2.2) and the above we deduce that

$$\begin{aligned}\sum_{k=0}^{[p/3]}\binom{3k}{k}a^k &\equiv -\Big(\frac p3\Big)(3a)^{\frac{p-(\frac p3)}{3}}U_{\frac{p-(\frac p3)}{3}+(\frac p3)}(9a,3a)\\ &= -\Big(\frac p3\Big)(3a)^{\frac{p-(\frac p3)}{3}}\cdot\frac{1}{2(3a)^{(1-(\frac p3))/2}}\Big(9aU_{\frac{p-(\frac p3)}{3}}(9a,3a)+\Big(\frac p3\Big)V_{\frac{p-(\frac p3)}{3}}(9a,3a)\Big)\\ &\equiv \Big(\frac p3\Big)(3a)^{\frac{p-(\frac p3)}{3}+\frac{(\frac p3)-1}{2}}\frac{1}{27a-4}\Big(\frac{-a}{p}\Big)(3a)^{\frac{p-(\frac p3)}{6}-1}(2x_0^2-9ax_0-12a)\\ &\equiv \frac{1}{3a(27a-4)}(2x_0^2-9ax_0-12a) \pmod p.\end{aligned}$$

As $x_0^3\equiv 9ax_0+27a^2 \pmod p$ we see that

$$(2x_0^2-9ax_0-12a)(2x_0+9a)\equiv 3a(4-27a)x_0 \pmod p.$$

Hence

$$\sum_{k=0}^{[p/3]}\binom{3k}{k}a^k\equiv\frac{1}{3a(27a-4)}(2x_0^2-9ax_0-12a)\equiv-\frac{x_0}{2x_0+9a} \pmod p.$$

Set $x=-\frac{x_0}{2x_0+9a}$. Then $x_0=-\frac{9ax}{2x+1}$ and

$$\begin{aligned}&(27a-4)x^3+3x+1\\ &=(4-27a)\frac{x_0^3}{(2x_0+9a)^3}-\frac{3x_0}{2x_0+9a}+1=-\frac{27a(x_0^3-9ax_0-27a^2)}{(2x_0+9a)^3}.\end{aligned}$$

As $X\equiv x_0 \pmod p$ is the unique solution of $X^3-9aX-27a^2\equiv 0 \pmod p$ we see that $X\equiv x\equiv\sum_{k=0}^{[p/3]}\binom{3k}{k}a^k \pmod p$ is the unique solution of $(27a-4)X^3+3X+1\equiv 0 \pmod p$. This proves the theorem.

Remark 3.2. Let $p>3$ be a prime. By Theorems 3.8-3.10 we have:

$$\begin{aligned}&\sum_{k=1}^{[p/3]}\binom{3k}{k}\equiv 0 \pmod p\\ &\quad\iff p=x^2+xy+52y^2 \text{ or } p=8x^2+7xy+8y^2\ne 23,\\ &\sum_{k=1}^{[p/3]}\binom{3k}{k}(-1)^k\equiv 0 \pmod p\end{aligned}$$

$$\iff p = x^2 + xy + 70y^2, 8x^2 + 3xy + 9y^2 \text{ or } p = 9x^2 + 9xy + 10y^2 \neq 31.$$

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References

[1] A. Adamchuk, Comments on OEIS A066796 in 2006, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A066796.

[2] D.H. Lehmer, An extended theory of Lucas' functions, Ann. Math. **31**(1930),419-448.

[3] Z.H. Sun, The combinatorial sum $\sum_{k=0,k\equiv r \pmod m}^{n} \binom{n}{k}$ and its applications in number theory I, J. Nanjing Univ. Math. Biquarterly **9**(1992), 227-240.

[4] Z.H. Sun, The combinatorial sum $\sum_{k=0,k\equiv r \pmod m}^{n} \binom{n}{k}$ and its applications in number theory II, J. Nanjing Univ. Math. Biquarterly **10**(1993), 105-118.

[5] Z.H. Sun, The combinatorial sum $\sum_{k\equiv r \pmod m} \binom{n}{k}$ and its applications in number theory III, J. Nanjing Univ. Math. Biquarterly **12**(1995), 90-102.

[6] Z.H. Sun, On the theory of cubic residues and nonresidues, Acta Arith. **84**(1998), 291-335.

[7] Z.H. Sun, Values of Lucas sequences modulo primes, Rocky Mountain J. Math. **33**(2003), 1123-1145.

[8] Z.H. Sun, Cubic and quartic congruences modulo a prime, J. Number Theory **102**(2003), 41-89.

[9] Z.H. Sun, Cubic residues and binary quadratic forms, J. Number Theory **124**(2007), 62-104.

[10] Z.H. Sun, On the quadratic character of quadratic units, J. Number Theory **128**(2008),1295-1335.

[11] Z.H. Sun, Congruences concerning Legendre polynomials, Proc. Amer. Math. Soc. **139**(2011), 1915-1929.

[12] Z.H. Sun, Congruences involving $\binom{2k}{k}^2\binom{3k}{k}$, J. Number Theory **133**(2013), 1572-1595.

[13] Z.H. Sun, Congruences concerning Legendre polynomials II, J. Number Theory **133**(2013), 1950-1976.

[14] Z.H. Sun and Z.W. Sun, Fibonacci numbers and Fermat's last theorem, Acta Arith. **60**(1992), 371-388.

[15] Z.W. Sun, Various congruences involving binomial coefficients and higher-order Catalan numbers, arXiv:0909.3808v2, http://arxiv.org/abs/0909.3808.

[16] Z. W. Sun and R. Tauraso, New congruences for central binomial coefficients, Adv. in Appl. Math. **45**(2010), 125-148.

[17] H.C. Williams, Édouard Lucas and Primality Testing, Canadian Mathematical Society Series of Monographs and Advanced Texts, Vol.22, Wiley, New York, 1998, pp. 74-92.

[18] L.L. Zhao, H. Pan and Z.W. Sun, Some congruences for the second-order Catalan numbers, Proc. Amer. Math. Soc. **138**(2010), 37-46.