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Supercongruences involving products of three binomial coefficients

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Abstract

Let $p>3$ be a prime, and let a be a rational p-adic integer. Using the WZ method we establish the congruences for $\sum_{k=0}^{p-1}\binom ak\binom{-1-a}k\binom{2k}k\frac{w(k)}{4^k}$ modulo p^3, where $w(k)\in\{1,\frac 1{k+1},\frac 1{(k+1)^2},\frac 1{2k-1}\}$. Taking $a=-\frac 12,-\frac 13,-\frac 14,-\frac 16$ in the congruences confirms some conjectures posed by the author earlier.

Keywords: congruence; binomial coefficient; combinatorial identity; Euler number; binary quadratic form
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1. Introduction

The generalized binomial coefficient $\binom ak$ is given by

$$\binom a0=1,\quad \binom a{-k}=0\quad\text{and}\quad \binom ak=\frac{a(a-1)\cdots(a-k+1)}{k!}\quad\text{for}\quad k=1,2,3,\ldots$$

For $a\in\mathbb Z$ and given odd prime p let $(\frac ap)$ denote the Legendre symbol. For positive integers a,b and n, if $n=ax^2+by^2$ for some integers x and y, we briefly write that $n=ax^2+by^2$. Let p be an odd prime. In 1987, Beukers[3] conjectured a congruence equivalent to

$$\sum_{k=0}^{p-1}\frac{\binom{2k}k^3}{64^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p=x^2+4y^2\equiv 1\pmod 4,\\ 0\pmod{p^2}&\text{if }p\equiv 3\pmod 4.\end{cases}$$

This congruence was proved by several mathematicians including Ishikawa[5]($p\equiv 1\pmod 4$), Van Hamme[30]($p\equiv 3\pmod 4$) and Ahlgren[1]. Combining the results in [7], [21] and [29], in [25] the author stated that

$$(1.1)\quad \sum_{k=0}^{p-1}\frac{\binom{2k}k^3}{64^k}\equiv\begin{cases}4x^2-2p-\dfrac{p^2}{4x^2}\pmod{p^3}&\text{if }p=x^2+4y^2\equiv 1\pmod 4,\\ -\dfrac{p^2}4\binom{\frac{p-3}2}{\frac{p-3}4}^{-2}\equiv -p^2\binom{\frac{p-1}2}{\frac{p-3}4}^{-2}\pmod{p^3}&\text{if }p\equiv 3\pmod 4.\end{cases}$$

Let $p>3$ be a prime. In 2003, Rodriguez-Villegas[11] posed 22 conjectures on supercongruences modulo p^2. In particular, the following congruences are equivalent to certain conjectures due to Rodriguez-Villegas:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+3y^2\equiv 1\pmod 3,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8,\end{cases}$$
$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

These conjectures have been solved by Mortenson[9] and Sun[26]. In 2018, Liu[6] conjectured the congruences modulo p^3 for

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\quad\text{and}\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}$$

in terms of p-adic Gamma functions. In [22], the author conjectured that

$$(1.2)\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } p=x^2+3y^2\equiv 1\pmod 3,\\ -\frac{p^2}{2}\binom{(p-1)/2}{(p-5)/6}^{-2}\pmod{p^3} & \text{if } p\equiv 2\pmod 3,\end{cases}$$

$$(1.3)\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ \frac{p^2}{3}\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 8,\\ -\frac{3}{2}p^2\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod 8,\end{cases}$$

$$(1.4)\quad \left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\ \frac{5}{12}p^2\binom{(p-3)/2}{(p-3)/4}^{-2}\pmod{p^3} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

where $[a]$ is the greatest integer not exceeding a. It is easy to see that (see [15-18])

$$\binom{-\frac12}{k}=\frac{\binom{2k}{k}}{(-4)^k},\quad \binom{-\frac13}{k}\binom{-\frac23}{k}=\frac{\binom{2k}{k}\binom{3k}{k}}{27^k},$$
$$\binom{-\frac14}{k}\binom{-\frac34}{k}=\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k},\quad \binom{-\frac16}{k}\binom{-\frac56}{k}=\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}.$$

Thus, a natural and general problem is to determine $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k}$ modulo p^3, where $p>3$ is a prime and a is a rational p-adic integer.

For a prime p let $\mathbb{Z}_p$ be the set of rational numbers whose denominators are not divisible by p. For $a\in\mathbb{Z}_p$ let $\langle a\rangle_p$ be given by $\langle a\rangle_p\in\{0,1,\dots,p-1\}$ and $a\equiv\langle a\rangle_p\pmod p$. In

[18], the author showed that for any odd prime p and $a \in \mathbb{Z}_p$,

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k} \equiv 0 \pmod{p^2} \quad \text{for} \quad \langle a\rangle_p \equiv 1 \pmod{2}.$$

For an odd prime p and $x \in \mathbb{Z}_p$, the p-adic Gamma function $\Gamma_p(x)$ is defined by

$$\Gamma_p(0)=1,\ \Gamma_p(n)=(-1)^n \prod_{\substack{k\in\{1,2,\ldots,n-1\}\\ p\nmid k}} k \text{ for } n=1,2,3,\ldots,\ \Gamma_p(x)=\lim_{\substack{n\in\{0,1,\ldots\}\\ |x-n|_p\to 0}}\Gamma_p(n).$$

In [10, Theorem 5.2], Pan, Tauraso and Wang stated a result equivalent to

$$\sum_{k=0}^{p-1}\frac{\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}}{4^k} \equiv \begin{cases} \dfrac{\Gamma_p(\frac12)^2}{\Gamma_p(\frac{2+a}{2})^2\Gamma_p(\frac{1-a}{2})^2} \pmod{p^3} & \text{if } 2\mid \langle a\rangle_p,\\ \dfrac{a'(a'+1)p^2\Gamma_p(\frac12)^2}{4\Gamma_p(\frac{2+a}{2})^2\Gamma_p(\frac{1-a}{2})^2} \pmod{p^3} & \text{if } 2\nmid \langle a\rangle_p,\end{cases} \tag{1.5}$$

where $a' = (a-\langle a\rangle_p)/p$. However, they only gave a sketch of the proof in the case $2 \mid \langle a\rangle_p$, and their method is complicated and based on hypergeometric series identities. It should be mentioned that (1.5) was first conjectured by Liu [6] in the case $2 \mid \langle a\rangle_p$. Using (1.5) and Jacobi sums, recently Mao [8] proved (1.2) and (1.3). We also note that Guo [4] established three congruences modulo p^3 concerning the sums in (1.2)-(1.4) via q-congruences. For example, Guo showed that for any prime $p \equiv 5 \pmod 6$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv \frac{2p\binom{-2/3}{(2p-1)/3}}{3\binom{-7/6}{(2p-1)/3}} + \frac{p\binom{-5/6}{(p-2)/3}}{3\binom{-4/3}{(p-2)/3}} \pmod{p^3}.$$

In [22], [24] and [25], the author posed numerous conjectures on the congruences modulo p^3 for the sums

$$\sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}^3}{m^k},\ \sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}^2\binom{3k}{k}}{m^k},\ \sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}^2\binom{4k}{2k}}{m^k} \quad\text{and}\quad \sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k},$$

where $w(k) \in \{1, \frac{1}{k+1}, \frac{1}{2k-1}, \frac{1}{(k+1)^2}, k, k^2, k^3\}$ and m is an integer not divisible by p.

Let $p>3$ be a prime and $a \in \mathbb{Z}_p$. Inspired by the above work, using the WZ method we establish the congruences modulo p^3 for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{w(k)}{4^k}$ in terms of harmonic numbers, where $w(k) \in \{1, \frac{1}{k+1}, \frac{1}{(k+1)^2}, \frac{1}{2k-1}\}$. Our approach is natural and elementary, and the result in the case $w(k)=1$ seems better than (1.5) since $\Gamma_p(x)$ is more complicated than harmonic numbers and it is very difficult to determine $\Gamma_p(x)$ modulo p^3. On the other hand, the proof of (1.5) depends on hypergeometric series identities. As consequences, taking $a = -\frac12, -\frac13, -\frac14, -\frac16$ in our main results we obtain explicit congruences for

$$\sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}^3}{64^k},\ \sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}^2\binom{3k}{k}}{108^k},\ \sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\ \sum_{k=0}^{p-1}\frac{w(k)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}$$

modulo p^3 or p^2 and so solve some conjectures in [22] and [25].

The Catalan numbers $\{C_k\}$ are defined by $C_k=\binom{2k}{k}\frac{1}{k+1}$ $(k\ge 0)$. For convenience we also define $C_{-1}=-\frac12$. Then clearly $\binom{2k}{k}\frac{1}{2k-1}=2C_{k-1}$ for $k\ge 0$. As typical results in the paper, for any prime $p\equiv 1 \pmod 4$ and so $p=x^2+4y^2$ we have

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2C_k}{64^k}\equiv 4x^2-2p \pmod{p^3},\quad \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k^2}{64^k}\equiv 8x^2-4p+\frac{p^2}{x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2C_{k-1}}{64^k}\equiv -x^2+\frac p2+\frac{p^2}{8x^2} \pmod{p^3},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}C_{k-1}^2}{64^k}\equiv \frac{x^2}{2}-\frac p4-\frac{p^2}{8x^2} \pmod{p^3},$$

for any prime $p\equiv 1 \pmod 3$ and so $p=x^2+3y^2$ we have

$$\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k}\equiv 4x^2-2p \pmod{p^3},\ \sum_{k=0}^{p-2}\frac{\binom{3k}{k}C_k^2}{108^k}\equiv 8x^2-4p+\frac{9p^2}{8x^2} \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}C_{k-1}}{108^k}\equiv -\frac{10}{9}x^2+\frac59p+\frac{p^2}{8x^2} \pmod{p^3}.$$

In addition to the above notation, throughout this paper we use the following notations. Let $H_0=H_0^{(2)}=0$. For $n\ge 1$ let $H_n=1+\frac12+\cdots+\frac1n$ and $H_n^{(2)}=1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}$. For an odd prime p and $a\in\mathbb{Z}_p$ set $q_p(a)=(a^{p-1}-1)/p$ and

$$R_1(p)=(2p+2-2^{p-1})\binom{(p-1)/2}{[p/4]}^2,$$

$$R_2(p)=(5-4(-1)^{\frac{p-1}{2}})\Big(1+(4+2(-1)^{\frac{p-1}{2}})p-4(2^{p-1}-1)-\frac p2H_{[\frac p8]}\Big)\binom{(p-1)/2}{[p/8]}^2,$$

$$R_3(p)=\Big(1+2p+\frac43(2^{p-1}-1)-\frac32(3^{p-1}-1)\Big)\binom{(p-1)/2}{[p/6]}^2.$$

Define

$$S_n(a)=\sum_{k=0}^{n-1}\binom ak\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k}\quad (n=1,2,3,\ldots).$$

Let $\{E_n\}$ be the Euler numbers given by

$$E_{2n-1}=0,\quad E_0=1,\quad E_{2n}=-\sum_{k=1}^{n}\binom{2n}{2k}E_{2n-2k}\ (n=1,2,3,\ldots),$$

and the sequence $\{U_n\}$ be given by

$$U_{2n-1}=0,\quad U_0=1,\quad U_{2n}=-2\sum_{k=1}^{n}\binom{2n}{2k}U_{2n-2k}\ (n=1,2,3,\ldots).$$

2. The congruence for $\sum_{k=0}^{p-1}\binom ak\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k}$ modulo p^3

For $k=0,1,2,\ldots$ set

$$F(a,k)=\binom ak\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k}$$

and

$$G(a,k)=(a+2)(2a+3)\frac{k}{4^{k-1}(a+1+k)}\binom{2k-1}{k-1}\binom{a+1}{k-1}\binom{-3-a}{k-1}.$$

It is easy to check that

$$(a+2)^2F(a+2,k)-(a+1)^2F(a,k)=G(a,k+1)-G(a,k).$$

Thus,

$$\begin{aligned}&(a+2)^2S_n(a+2)-(a+1)^2S_n(a)\\&=(a+2)^2\sum_{k=0}^{n-1}F(a+2,k)-(a+1)^2\sum_{k=0}^{n-1}F(a,k)\\&=\sum_{k=0}^{n-1}(G(a,k+1)-G(a,k))=G(a,n)-G(a,0)=G(a,n).\end{aligned}$$

That is,

$$\begin{aligned}&(a+2)^2S_n(a+2)-(a+1)^2S_n(a)\\&=(a+2)(2a+3)\frac{n}{4^{n-1}(a+1+n)}\binom{2n-1}{n-1}\binom{a+1}{n-1}\binom{-3-a}{n-1}.\end{aligned}\tag{2.1}$$

Lemma 2.1. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $a\not\equiv-1\pmod p$ and $a'=(a-\langle a\rangle_p)/p$. Then*

$$\begin{aligned}&(a+2)^2S_p(a+2)-(a+1)^2S_p(a)\\&\equiv\begin{cases}(\frac{1}{a+1}+\frac{1}{a+2})a'(a'+1)p^3\pmod{p^4} & \text{if }\langle a\rangle_p<p-2,\\(a+2)p\pmod{p^3} & \text{if }\langle a\rangle_p=p-2.\end{cases}\end{aligned}$$

Proof. From (2.1) we see that

$$\begin{aligned}&(a+2)^2S_p(a+2)-(a+1)^2S_p(a)\\&=(a+2)(2a+3)\frac{p}{4^{p-1}(a+1+p)}\binom{2p-1}{p}\binom{a+2-1}{p-1}\binom{-a-2-1}{p-1}.\end{aligned}$$

We first assume $\langle a\rangle_p<p-2$. Then $a+2-\langle a+2\rangle_p=a-\langle a\rangle_p=a'p$. From [20, Lemma 2.2] we know that

$$\binom{a+2-1}{p-1}\binom{-a-2-1}{p-1}\equiv\frac{a'(a'+1)p^2}{(a+2)^2}\pmod{p^3}.\tag{2.2}$$

Hence,

$$\begin{aligned}&(a+2)^2S_p(a+2)-(a+1)^2S_p(a)\\&\equiv(a+2)(2a+3)\frac{p}{4^{p-1}(a+1+p)}\binom{2p-1}{p-1}\frac{a'(a'+1)p^2}{(a+2)^2}\\&\equiv\Big(\frac{1}{a+1}+\frac{1}{a+2}\Big)a'(a'+1)p^3\pmod{p^4}.\end{aligned}$$

Now we assume that $\langle a\rangle_p = p-2$. Then $a+2 = p(a'+1)$. Appealing to [23, (2.6)],

$$\binom{a+2-1}{p-1}\binom{-a-2-1}{p-1} = \binom{p-1+a'p}{p-1}\binom{p-1-(a'+2)p}{p-1}$$
$$\equiv (1+a'pH_{p-1})(1-(a'+2)pH_{p-1}) \equiv 1 \pmod{p^2}.$$

Hence,

$$(a+2)^2 S_p(a+2) - (a+1)^2 S_p(a)$$
$$\equiv (a+2)(2a+3)\frac{p}{4^{p-1}(a+1+p)}\binom{2p-1}{p-1} \equiv (a+2)p \pmod{p^3}.$$

This completes the proof.

Lemma 2.2. *Let p be an odd prime and $t \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{pt}{k}\binom{-1-pt}{k}\binom{2k}{k}\frac{1}{4^k} \equiv 1 - 2t(2^{p-1}-1) + (2t^2+t)(2^{p-1}-1)^2 \pmod{p^3}.$$

Proof. Clearly,

$$\sum_{k=0}^{p-1}\binom{pt}{k}\binom{-1-pt}{k}\binom{2k}{k}\frac{1}{4^k}$$
$$= 1 + \sum_{k=1}^{p-1}\frac{pt}{pt-k}\cdot\frac{((-1)^2-p^2t^2)((-2)^2-p^2t^2)\cdots((-k)^2-p^2t^2)}{k!^2}\cdot\frac{\binom{2k}{k}}{4^k}$$
$$\equiv 1 - \sum_{k=1}^{p-1}\frac{pt(pt+k)}{k^2}\cdot\frac{\binom{2k}{k}}{4^k} \pmod{p^3}.$$

By [19, Remark 3.1] or [27],

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k\cdot 4^k} = \sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{-1/2}{k} \equiv 2q_p(2) - pq_p(2)^2 \pmod{p^2}. \tag{2.3}$$

Taking $x = \frac{1}{4}$ in [28, (9)] and then applying [13, Theorem 4.1] we see that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2\cdot 4^k} \equiv 4\sum_{k=1}^{p-1}\frac{1}{k^2\cdot 2^k} \equiv -2q_p(2)^2 \pmod{p}. \tag{2.4}$$

It then follows that

$$\sum_{k=0}^{p-1}\binom{pt}{k}\binom{-1-pt}{k}\binom{2k}{k}\frac{1}{4^k} \equiv 1 - pt(2q_p(2) - pq_p(2)^2) - p^2t^2(-2q_p(2)^2) \pmod{p^3},$$

which yields the result.

Lemma 2.3. *Let p be an odd prime, $n \in \{1, 2, \ldots, \frac{p-1}{2}\}$ and $t \in \mathbb{Z}_p$. Then*

$$\binom{\frac{p-1}{2}+pt}{n}$$

$$\equiv \binom{\frac{p-1}{2}}{n}\Big(1-2pt\sum_{k=1}^{n}\frac{1}{2k-1}+2p^2t\Big(t\Big(\sum_{k=1}^{n}\frac{1}{2k-1}\Big)^2-(t+1)\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\Big)\Big) \pmod{p^3}.$$

Proof. For $m \in \{n, n+1, \dots, p-1\}$ we see that

(2.5)

$$\begin{aligned}\binom{m+pt}{n} &= \frac{(pt+m)(pt+m-1)\cdots(pt+m-(n-1))}{n!}\\ &\equiv \binom{m}{n}\Big(1+pt\sum_{m-n+1\le k\le m}\frac{1}{k}+p^2t^2\sum_{m-n+1\le i<j\le m}\frac{1}{ij}\Big)\\ &= \binom{m}{n}\Big(1+pt(H_m-H_{m-n})+\frac{1}{2}p^2t^2\big((H_m-H_{m-n})^2-(H_m^{(2)}-H_{m-n}^{(2)})\big)\Big) \pmod{p^3}.\end{aligned}$$

For given positive integer r we have

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}\frac{1}{k^r}-\sum_{k=1}^{(p-1)/2-n}\frac{1}{k^r}\\ &=\sum_{k=1}^{n}\frac{1}{\big(\frac{p-(2k-1)}{2}\big)^r}=2^r\sum_{k=1}^{n}\frac{(p+2k-1)^r}{(p^2-(2k-1)^2)^r}\equiv(-2)^r\sum_{k=1}^{n}\frac{(2k-1)^r+rp(2k-1)^{r-1}}{(2k-1)^{2r}}\\ &=(-2)^r\sum_{k=1}^{n}\frac{1}{(2k-1)^r}+(-2)^r rp\sum_{k=1}^{n}\frac{1}{(2k-1)^{r+1}} \pmod{p^2}.\end{aligned}$$

Hence,

$$\begin{aligned}H_{\frac{p-1}{2}}-H_{\frac{p-1}{2}-n}&\equiv -2\sum_{k=1}^{n}\frac{1}{2k-1}-2p\sum_{k=1}^{n}\frac{1}{(2k-1)^2} \pmod{p^2},\\ H^{(2)}_{\frac{p-1}{2}}-H^{(2)}_{\frac{p-1}{2}-n}&\equiv 4\sum_{k=1}^{n}\frac{1}{(2k-1)^2} \pmod{p}.\end{aligned}$$

Now, from the above we deduce that

$$\begin{aligned}\binom{\frac{p-1}{2}+pt}{n}\equiv\binom{\frac{p-1}{2}}{n}&\Big(1+pt\Big(-2\sum_{k=1}^{n}\frac{1}{2k-1}-2p\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\Big)\\ &+\frac{1}{2}p^2t^2\Big(\Big(-2\sum_{k=1}^{n}\frac{1}{2k-1}\Big)^2-4\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\Big)\Big) \pmod{p^3},\end{aligned}$$

which yields the result.

Theorem 2.1. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0,-1 \pmod p$ and $a'=(a-\langle a\rangle_p)/p$. If $2\mid\langle a\rangle_p$ and $\langle a\rangle_p=2n$, then*

$$\begin{aligned}S_p(a)&\equiv\frac{\binom{(a-1)/2}{n}^2}{\binom{a/2}{n}^2}\big(1-2a'(2^{p-1}-1)+a'(2a'+1)(2^{p-1}-1)^2\big)\\ &\equiv\binom{(p-1)/2}{n}^2\Big(1+p\big((2a'+2)H_{2n}-(2a'+1)H_n-2a'q_p(2)\big)\end{aligned}$$

$$+\frac{p^2}{2}\Big(2a'q_p(2)^2+\big((2a'+2)H_{2n}-(2a'+1)H_n-2a'q_p(2)\big)^2$$
$$+\frac{1}{2}(2a'^2-1)H_n^{(2)}+2(1-a'^2)H_{2n}^{(2)}\Big)\Big)\pmod{p^3}.$$

If $2\nmid\langle a\rangle_p$, then

$$S_p(a)\equiv\frac{4^{\langle a\rangle_p-1}\cdot a'(a'+1)p^2}{a^2\binom{\langle a\rangle_p-1}{\frac{\langle a\rangle_p-1}{2}}^2}\equiv\frac{a'(a'+1)p^2}{a^2\binom{(p-1)/2}{(\langle a\rangle_p-1)/2}^2}\pmod{p^3}.$$

Proof. We first assume that $2\mid\langle a\rangle_p$ and $\langle a\rangle_p=2n$. From Lemma 2.1 we see that

$$\begin{aligned}S_p(a)&\equiv\frac{(a-1)^2}{a^2}S_p(a-2)\equiv\frac{(a-1)^2}{a^2}\cdot\frac{(a-3)^2}{(a-2)^2}S_p(a-4)\equiv\cdots\\&\equiv\frac{(a-1)^2(a-3)^2\cdots(a-2n+1)^2}{a^2(a-2)^2\cdots(a-2n+2)^2}S_p(a-2n)\\&=\prod_{k=1}^n\frac{(\frac{a+1}{2}-k)^2}{(\frac{a+2}{2}-k)^2}\cdot S_p(a-2n)=\frac{\binom{(a-1)/2}{n}^2}{\binom{a/2}{n}^2}S_p(a-2n)\pmod{p^3}.\end{aligned}$$

By Lemma 2.2,

$$S_p(a-2n)=S_p(a'p)\equiv1-2a'(2^{p-1}-1)+a'(2a'+1)(2^{p-1}-1)^2\pmod{p^3}.$$

Thus,

$$S_p(a)\equiv\binom{(a-1)/2}{n}^2\binom{a/2}{n}^{-2}\big(1-2a'pq_p(2)+a'(2a'+1)p^2q_p(2)^2\big)\pmod{p^3}.$$

Since $(1+bp+cp^2)(1-bp+(b^2-c)p^2)\equiv1\pmod{p^3}$, appealing to (2.5) we get

$$\begin{aligned}\binom{a/2}{n}^{-1}\equiv\binom{n+a'p/2}{n}^{-1}&\equiv\Big(1+\frac{1}{2}a'pH_n+\frac{1}{8}a'^2p^2(H_n^2-H_n^{(2)})\Big)^{-1}\\&\equiv1-\frac{1}{2}a'pH_n+\frac{1}{8}a'^2p^2(H_n^2+H_n^{(2)})\pmod{p^3}.\end{aligned}$$

By Lemma 2.3,

$$\begin{aligned}\binom{\frac{a-1}{2}}{n}&=(-1)^n\binom{-\frac{a-1}{2}+n-1}{n}=(-1)^n\binom{\frac{p-1}{2}-\frac{a'+1}{2}p}{n}\\&\equiv(-1)^n\binom{\frac{p-1}{2}}{n}\Big(1+(a'+1)p\sum_{k=1}^n\frac{1}{2k-1}\\&\quad+(a'+1)p^2\Big(\frac{a'+1}{2}\Big(\sum_{k=1}^n\frac{1}{2k-1}\Big)^2+\frac{1-a'}{2}\sum_{k=1}^n\frac{1}{(2k-1)^2}\Big)\Big)\pmod{p^3}.\end{aligned}$$

Therefore,

$$(-1)^n\binom{\frac{a-1}{2}}{n}\binom{\frac{a}{2}}{n}^{-1}\binom{\frac{p-1}{2}}{n}^{-1}$$

$$\equiv\Big(1+(a'+1)p\sum_{k=1}^n\frac{1}{2k-1}+(a'+1)p^2\Big(\frac{a'+1}{2}\Big(\sum_{k=1}^n\frac{1}{2k-1}\Big)^2+\frac{1-a'}{2}\sum_{k=1}^n\frac{1}{(2k-1)^2}\Big)\Big)$$
$$\times\Big(1-\frac12a'pH_n+\frac18a'^2p^2(H_n^2+H_n^{(2)})\Big)$$
$$\equiv 1+p\Big((a'+1)\sum_{k=1}^n\frac{1}{2k-1}-\frac{a'}{2}H_n\Big)+p^2\Big(\frac18a'^2H_n^2+\frac18a'^2H_n^{(2)}$$
$$+\frac{(a'+1)^2}{2}\Big(\sum_{k=1}^n\frac{1}{2k-1}\Big)^2+\frac{1-a'^2}{2}\sum_{k=1}^n\frac{1}{(2k-1)^2}-\frac12a'(a'+1)H_n\sum_{k=1}^n\frac{1}{2k-1}\Big)$$
$$=1+p\Big((a'+1)\sum_{k=1}^n\frac{1}{2k-1}-\frac{a'}{2}H_n\Big)$$
$$+p^2\Big(\frac18\Big(a'H_n-2(a'+1)\sum_{k=1}^n\frac{1}{2k-1}\Big)^2+\frac18a'^2H_n^{(2)}+\frac{1-a'^2}{2}\sum_{k=1}^n\frac{1}{(2k-1)^2}\Big)\pmod{p^3}.$$

Note that $(1+bp+cp^2)^2\equiv 1+2bp+(b^2+2c)p^2 \pmod{p^3}$. From the above we derive that

$$\binom{\frac{a-1}{2}}{n}^2\binom{\frac a2}{n}^{-2}\binom{\frac{p-1}{2}}{n}^{-2}$$
$$\equiv 1+p\Big(2(a'+1)\sum_{k=1}^n\frac{1}{2k-1}-a'H_n\Big)+p^2\Big(2\Big((a'+1)\sum_{k=1}^n\frac{1}{2k-1}-\frac{a'}{2}H_n\Big)^2$$
$$+\frac14a'^2H_n^{(2)}+(1-a'^2)\sum_{k=1}^n\frac{1}{(2k-1)^2}\Big)$$
$$=1+p\Big(2(a'+1)(H_{2n}-\frac12H_n)-a'H_n\Big)+p^2\Big(2\Big((a'+1)(H_{2n}-\frac12H_n)-\frac{a'}{2}H_n\Big)^2$$
$$+\frac14a'^2H_n^{(2)}+(1-a'^2)\big(H_{2n}^{(2)}-\frac14H_n^{(2)}\big)\Big)$$
$$=1+p((2a'+2)H_{2n}-(2a'+1)H_n)+p^2\Big(\frac12((2a'+2)H_{2n}-(2a'+1)H_n)^2$$
$$+(1-a'^2)H_{2n}^{(2)}+\frac{2a'^2-1}{4}H_n^{(2)}\Big)\pmod{p^3}$$

and so

$$S_p(a)\equiv\binom{(p-1)/2}{n}^2(1-2a'pq_p(2)+a'(2a'+1)p^2q_p(2)^2)$$
$$\times\Big(1+p((2a'+2)H_{2n}-(2a'+1)H_n)+p^2\Big(\frac12((2a'+2)H_{2n}-(2a'+1)H_n)^2$$
$$+(1-a'^2)H_{2n}^{(2)}+\frac{2a'^2-1}{4}H_n^{(2)}\Big)\Big)$$
$$\equiv\binom{(p-1)/2}{n}^2\Big(1+p(-2a'q_p(2)+2(a'+1)H_{2n}-(2a'+1)H_n)+p^2\Big(a'q_p(2)^2$$
$$+\frac12\big((2a'+2)H_{2n}-(2a'+1)H_n-2a'q_p(2)\big)^2$$
$$+(1-a'^2)H_{2n}^{(2)}+\frac{2a'^2-1}{4}H_n^{(2)}\Big)\Big)\pmod{p^3},$$

which yields the result in the case $2 \mid \langle a\rangle_p$.

Now assume that $2 \nmid \langle a\rangle_p$. By Lemma 2.1, for $\langle a\rangle_p \le p-4$,

$$\begin{aligned}
S_p(a) &\equiv \frac{(a+2)^2}{(a+1)^2} S_p(a+2) \equiv \frac{(a+2)^2}{(a+1)^2}\cdot\frac{(a+4)^2}{(a+3)^2} S_p(a+4) \equiv \cdots \\
&\equiv \prod_{k=0}^{\frac{p-\langle a\rangle_p}{2}-2} \frac{(a+2k+2)^2}{(a+2k+1)^2}\cdot S_p(p+a-\langle a\rangle_p-2) \\
&= \prod_{k=0}^{\frac{p-\langle a\rangle_p}{2}-2} \frac{(a+2k+1)^2(a+2k+2)^2}{(a+2k+1)^4}\cdot S_p(p+a-\langle a\rangle_p-2) \\
&= \frac{(a+1)^2(a+2)^2\cdots(p+a-\langle a\rangle_p-3)^2(p+a-\langle a\rangle_p-2)^2}{2^{4(\frac{p-\langle a\rangle_p}{2}-1)}(\frac{a+1}{2}(\frac{a+1}{2}+1)\cdots(\frac{a+1}{2}+\frac{p-\langle a\rangle_p}{2}-2))^4} \\
&\quad \times S_p(p+a-\langle a\rangle_p-2) \pmod{p^3}.
\end{aligned}$$

By Lemmas 2.1 and 2.2,

$$\begin{aligned}
((a'+1)p-1)^2 S_p((a'+1)p-2) &\equiv (a'+1)^2p^2 S_p((a'+1)p) - (a'+1)p^2 \\
&\equiv (a'+1)^2p^2 - (a'+1)p^2 = a'(a'+1)p^2 \pmod{p^3}.
\end{aligned}$$

Hence,

$$S_p(p+a-\langle a\rangle_p-2) = S_p((a'+1)p-2) \equiv a'(a'+1)p^2 \pmod{p^3}.$$

Now, from the above we deduce that

$$\begin{aligned}
S_p(a) &\equiv a'(a'+1)p^2\cdot\frac{(\langle a\rangle_p+1)^2(\langle a\rangle_p+2)^2\cdots(p-3)^2(p-2)^2}{2^{4(\frac{p-\langle a\rangle_p}{2}-1)}(\frac{\langle a\rangle_p+1}{2}(\frac{\langle a\rangle_p+1}{2}+1)\cdots(\frac{p-3}{2}))^4} \\
&= a'(a'+1)p^2\cdot\frac{(p-2)!^2\cdot\frac{\langle a\rangle_p-1}{2}!^4}{\langle a\rangle_p!^2\cdot 2^{2(p-\langle a\rangle_p)-4}\cdot\frac{p-3}{2}!^4} \\
&= a'(a'+1)p^2\cdot\frac{(\frac{p-1}{2})^4}{(p-1)^2}\binom{p-1}{\frac{p-1}{2}}^2\cdot\frac{1}{2^{2(p-\langle a\rangle_p)-4}\cdot(\frac{\langle a\rangle_p+1}{2})^2\binom{\langle a\rangle_p}{\frac{\langle a\rangle_p-1}{2}}^2} \\
&\equiv \frac{a'(a'+1)p^2}{2^{2(p-\langle a\rangle_p)}(\frac{a+1}{2})^2\binom{\langle a\rangle_p}{\frac{\langle a\rangle_p-1}{2}}^2} \equiv \frac{4^{\langle a\rangle_p}\cdot a'(a'+1)p^2}{(a+1)^2\binom{\langle a\rangle_p}{\frac{\langle a\rangle_p-1}{2}}^2} \\
&\equiv \frac{4^{\langle a\rangle_p-1}\cdot a'(a'+1)p^2}{a^2\binom{\langle a\rangle_p-1}{\frac{\langle a\rangle_p-1}{2}}^2} = \frac{a'(a'+1)p^2}{a^2\binom{-1/2}{(\langle a\rangle_p-1)/2}^2} \equiv \frac{a'(a'+1)p^2}{a^2\binom{(p-1)/2}{(\langle a\rangle_p-1)/2}^2} \pmod{p^3}.
\end{aligned}$$

For $\langle a\rangle_p = p-2$, using Lemmas 2.1 and 2.2 we see that

$$\begin{aligned}
S_p(a) &\equiv \frac{1}{(a+1)^2}\big((a+2)^2S_p(a+2)-(a+2)p\big) \equiv (a+2)^2-(a+2)p \\
&\equiv \frac{4^{\langle a\rangle_p-1}\cdot a'(a'+1)p^2}{a^2\binom{\langle a\rangle_p-1}{\frac{\langle a\rangle_p-1}{2}}^2} \equiv \frac{a'(a'+1)p^2}{a^2\binom{(p-1)/2}{(\langle a\rangle_p-1)/2}^2} \pmod{p^3}.
\end{aligned}$$

This completes the proof.

Corollary 2.1. *Let p be a prime with $p>3$ and $p\equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv\begin{cases}-\dfrac{5p^2}{\binom{(p-1)/2}{[p/12]}^2} \pmod{p^3} & \text{if } p\equiv 7 \pmod{12},\\ -\dfrac{p^2}{5\binom{(p-1)/2}{[p/12]}^2} \pmod{p^3} & \text{if } p\equiv 11 \pmod{12}.\end{cases}$$

Proof. Recall that $\binom{-\frac16}{k}\binom{-\frac56}{k}=\binom{3k}{k}\binom{6k}{3k}\frac{1}{432^k}$. Let $a=-\frac16$ or $-\frac56$ according as $p\equiv 7 \pmod{12}$ or $p\equiv 11 \pmod{12}$, and $a'=(a-\langle a\rangle_p)/p$. Then $\langle a\rangle_p=[\frac p6]$, $\frac{\langle a\rangle_p-1}{2}=[\frac{p}{12}]$, $a'=-\frac16$ and $a'(a'+1)=-\frac{5}{36}$. Now, the result follows from Theorem 2.1 immediately.

Lemma 2.4 (See [12, Theorem 5.2] and [13, Corollaries 3.3 and 3.7]). *Let $p>3$ be a prime. Then*

$$\begin{aligned}&H_{\frac{p-1}{2}}\equiv -2q_p(2)+pq_p(2)^2 \pmod{p^2},\quad H^{(2)}_{\frac{p-1}{2}}\equiv 0 \pmod p,\\&H_{[\frac p4]}\equiv -3q_p(2)+\frac32 pq_p(2)^2-(-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2},\\&H^{(2)}_{[\frac p4]}\equiv 4(-1)^{\frac{p-1}{2}}E_{p-3} \pmod p.\end{aligned}$$

We remark that putting $a=-\frac12$ in Theorem 2.1 and then applying Lemma 2.4 gives a natural and elementary proof of (1.1).

Lemma 2.5. *Let $p>3$ be a prime. Then*

$$S_p\Big(\frac12\Big)\equiv\begin{cases}-\dfrac{p^2}{4x^2} \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \dfrac{(p+1)^2}{2^{p-1}}\dbinom{(p-1)/2}{(p-3)/4}^2+\dfrac{p^2}{2}\dbinom{(p-1)/2}{(p-3)/4}^2E_{p-3} \pmod{p^3} & \text{if } 4\mid p-3.\end{cases}$$

Proof. Set $a=\frac12$. Then $\langle a\rangle_p=\frac{p+1}{2}$ and $a'=-\frac12$. For $p=x^2+4y^2\equiv 1 \pmod 4$, since $2\nmid\langle a\rangle_p$ we have

$$S_p\Big(\frac12\Big)\equiv\frac{-\frac12\cdot\frac12\cdot p^2}{\frac14\binom{(p-1)/2}{(p-1)/4}^2}\equiv-\frac{p^2}{4x^2} \pmod{p^3}$$

by Theorem 2.1 and the known fact $\binom{(p-1)/2}{(p-1)/4}\equiv 2(-1)^{\frac{x-1}{2}}x \pmod p$. Now assume that $p\equiv 3 \pmod 4$. Then $\langle a\rangle_p=2\cdot\frac{p+1}{4}$. From Theorem 2.1 and Lemma 2.4 we deduce that

$$\begin{aligned}S_p\Big(\frac12\Big)&\equiv\binom{(p-1)/2}{(p+1)/4}^2\Big(1+p(H_{\frac{p+1}{2}}+q_p(2))\\&\quad+\frac{p^2}{2}\Big(-q_p(2)^2+(H_{\frac{p+1}{2}}+q_p(2))^2-\frac14H^{(2)}_{\frac{p+1}{4}}+\frac32H^{(2)}_{\frac{p+1}{2}}\Big)\Big)\\&\equiv\binom{(p-1)/2}{(p+1)/4}^2\Big(1+p\Big(\frac{2}{p+1}-2q_p(2)+pq_p(2)^2+q_p(2)\Big)\\&\quad+\frac{p^2}{2}\Big(-q_p(2)^2+\Big(\frac{2}{p+1}-2q_p(2)+pq_p(2)^2+q_p(2)\Big)^2\end{aligned}$$

$$-\frac{1}{4}\Big(\frac{16}{(p+1)^2}-4E_{p-3}\Big)+\frac{3}{2}\cdot\frac{4}{(p+1)^2}\Big)\Big)$$

$$\equiv\binom{(p-1)/2}{(p-3)/4}^2\Big(1+p(2-q_p(2))+p^2((1-q_p(2))^2+\frac{1}{2}E_{p-3})\Big)$$

$$\equiv\frac{(p+1)^2}{2^{p-1}}\binom{(p-1)/2}{(p-3)/4}^2+\frac{p^2}{2}\binom{(p-1)/2}{(p-3)/4}^2E_{p-3}\pmod{p^3}.$$

This proves the lemma.

Theorem 2.2 (See [25, Conjecture 5.4]). *Let p be a prime with $p>3$. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2C_{k-1}}{64^k}$$

$$\equiv\begin{cases}-x^2+\frac{p}{2}+\frac{p^2}{8x^2}\pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\ -\frac{1}{8}\Big(2+p^2E_{p-3}\Big)\frac{(p+1)^2}{2^{p-1}}\binom{(p-1)/2}{(p-3)/4}^2+\frac{p^2}{4\binom{(p-1)/2}{(p-3)/4}^2}\pmod{p^3} & \text{if } 4\mid p-3.\end{cases}$$

Proof. Note that

$$\binom{\frac12}{k}\binom{-\frac32}{k}=\frac{1+2k}{1-2k}\binom{-\frac12}{k}^2=\Big(-1-\frac{2}{2k-1}\Big)\binom{2k}{k}^2\frac{1}{16^k}. \tag{2.6}$$

We then have

$$2\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k(2k-1)}+\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}=-\sum_{k=0}^{p-1}\binom{\frac12}{k}\binom{-\frac32}{k}\binom{2k}{k}\frac{1}{4^k}=-S_p\Big(\frac12\Big). \tag{2.7}$$

Now, applying (1.1), Lemma 2.5 and the fact that $\binom{2k}{k}\frac{1}{2k-1}=2C_{k-1}$ yields the result.

3. The congruence for $\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{C_k}{4^k}$ modulo p^3

Lemma 3.1. *For any positive integer n and real number $a\neq 0$ we have*

$$\sum_{k=0}^{n-1}\binom{a}{k}\binom{-1-a}{k}\frac{C_k}{4^k}$$

$$=S_n(a)+\frac{a+1}{a}S_n(a+1)-\frac{a+1}{a}\binom{a}{n-1}\binom{-2-a}{n-1}\binom{2n-1}{n-1}\frac{1}{4^{n-1}}.$$

Proof. Set

$$F(a,k)=\Big(\binom{a}{k}\binom{-1-a}{k}\Big(\frac{1}{k+1}-1\Big)-\frac{a+1}{a}\binom{a+1}{k}\binom{-2-a}{k}\Big)\binom{2k}{k}\frac{1}{4^k},$$

$$G(a,k)=-\frac{a+1}{a}\binom{a}{k-1}\binom{-2-a}{k-1}\binom{2k-1}{k-1}\frac{1}{4^{k-1}}.$$

It is easy to check that $F(a,k)=G(a,k+1)-G(a,k)$. Thus,

$$\sum_{k=0}^{n-1}F(a,k)=\sum_{k=0}^{n-1}(G(a,k+1)-G(a,k))=G(a,n)-G(a,0)=G(a,n).$$

This yields the result.

Lemma 3.2. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0,-1 \pmod p$ and $a'=(a-\langle a\rangle_p)/p$. Then*

$$G(a,p)\equiv\frac{\binom{a}{p-1}\binom{-1-a}{p-1}\binom{2(p-1)}{p-1}}{4^{p-1}\cdot p}\equiv-\frac{a'(a'+1)}{a(a+1)}p^2 \pmod{p^3},$$

where $G(a,k)$ is given in the proof of Lemma 3.1.

Proof. By [20, Lemma 2.2] and Fermat's little theorem,

$$\binom{a}{p-1}\binom{-2-a}{p-1}\binom{2p-1}{p-1}\frac{1}{4^{p-1}}\equiv\frac{a'(a'+1)}{(a+1)^2}p^2 \pmod{p^3}. \tag{3.1}$$

Thus, $G(a,p)\equiv-\frac{a'(a'+1)}{a(a+1)}p^2 \pmod{p^3}$. On the other hand,

$$\begin{aligned}\frac{\binom{a}{p-1}\binom{-1-a}{p-1}\binom{2(p-1)}{p-1}}{4^{p-1}\cdot p}&=\frac{a+1}{(a+p)(2p-1)}\binom{a}{p-1}\binom{-2-a}{p-1}\frac{\binom{2p-1}{p-1}}{4^{p-1}}\\&\equiv-\frac{a+1}{a}\cdot\frac{a'(a'+1)}{(a+1)^2}p^2=-\frac{a'(a'+1)}{a(a+1)}p^2 \pmod{p^3}.\end{aligned}$$

Thus, the lemma is proved.

Theorem 3.1. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0,-1 \pmod p$ and $a'=(a-\langle a\rangle_p)/p$. Then*

$$\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{C_k}{4^k}\equiv S_p(a)+\frac{a+1}{a}S_p(a+1) \pmod{p^3}.$$

Thus, for $2\mid\langle a\rangle_p$ we have

$$\begin{aligned}&\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{C_k}{4^k}\\&\equiv\binom{(p-1)/2}{\langle a\rangle_p/2}^2\Big(1+p\big((2a'+2)H_{\langle a\rangle_p}-(2a'+1)H_{\frac{\langle a\rangle_p}{2}}-2a'q_p(2)\big)\\&\quad+\frac{p^2}{2}\Big(2a'q_p(2)^2+\big((2a'+2)H_{\langle a\rangle_p}-(2a'+1)H_{\frac{\langle a\rangle_p}{2}}-2a'q_p(2)\big)^2\\&\quad+\frac{1}{2}(2a'^2-1)H^{(2)}_{\frac{\langle a\rangle_p}{2}}+2(1-a'^2)H^{(2)}_{\langle a\rangle_p}\Big)\Big)+\frac{a'(a'+1)}{a(a+1)}p^2\binom{(p-1)/2}{\langle a\rangle_p/2}^{-2} \pmod{p^3};\end{aligned}$$

for $2\nmid\langle a\rangle_p$ and $\langle a\rangle_p\neq p-2$ we have

$$\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{C_k}{4^k}$$

$$\equiv \frac{a+1}{a}\binom{(p-1)/2}{(\langle a\rangle_p+1)/2}^2\Big(1+p\big((2a'+2)H_{\langle a\rangle_p+1}-(2a'+1)H_{\frac{\langle a\rangle_p+1}{2}}-2a'q_p(2)\big)$$
$$+\frac{p^2}{2}\Big(2a'q_p(2)^2+\big((2a'+2)H_{\langle a\rangle_p+1}-(2a'+1)H_{\frac{\langle a\rangle_p+1}{2}}-2a'q_p(2)\big)^2$$
$$+\frac{1}{2}(2a'^2-1)H^{(2)}_{\frac{\langle a\rangle_p+1}{2}}+2(1-a'^2)H^{(2)}_{\langle a\rangle_p+1}\Big)\Big)$$
$$+\frac{a'(a'+1)}{(a+1)^2}p^2\binom{(p-1)/2}{(\langle a\rangle_p+1)/2}^{-2} \pmod{p^3}.$$

Proof. Taking $n=p$ in Lemma 3.1 and then applying Lemma 3.2 gives

$$\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{C_k}{4^k}-S_p(a)-\frac{a+1}{a}S_p(a+1)$$
$$=-\binom{a}{p-1}\binom{-1-a}{p-1}\binom{2(p-1)}{p-1}\frac{1}{4^{p-1}p}-\frac{a+1}{a}\binom{a}{p-1}\binom{-2-a}{p-1}\binom{2p-1}{p-1}\frac{1}{4^{p-1}}$$
$$\equiv 0 \pmod{p^3}.$$

Now, applying Theorem 2.1 yields the remaining results.

Theorem 3.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2C_k}{64^k}\equiv\begin{cases}4x^2-2p \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{(p+1)^2}{2^{p-1}}\binom{(p-1)/2}{(p-3)/4}^2-p^2\Big(\binom{(p-1)/2}{(p-3)/4}^{-2} & \\ \quad+\frac{1}{2}\binom{(p-1)/2}{(p-3)/4}^2E_{p-3}\Big) \pmod{p^3} & \text{if } 4\mid p-3.\end{cases}$$

Proof. Note that $p\mid\binom{2k}{k}$ for $\frac{p}{2}<k<p$. Taking $a=-\frac{1}{2}$ in Theorem 3.1 gives

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2C_k}{64^k}\equiv\sum_{k=0}^{p-2}\frac{\binom{2k}{k}^2C_k}{64^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}-S_p\Big(\frac{1}{2}\Big) \pmod{p^3}.$$

Now, applying (1.1) and Lemma 2.5 yields the result.

Lemma 3.3. *Let $p>3$ be a prime. Then*

$$H_{[\frac{p}{3}]}\equiv-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2-p\Big(\frac{p}{3}\Big)U_{p-3} \pmod{p^2},$$
$$H_{[\frac{2p}{3}]}\equiv-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2+2p\Big(\frac{p}{3}\Big)U_{p-3} \pmod{p^2},$$
$$H^{(2)}_{[\frac{p}{3}]}\equiv-H^{(2)}_{[\frac{2p}{3}]}\equiv 3\Big(\frac{p}{3}\Big)U_{p-3} \pmod{p}.$$

Proof. The first congruence was given in [14, Theorem 3.2]. By [14, Theorem 3.2], $\sum_{k=1}^{[2p/3]}\frac{(-1)^{k-1}}{k}\equiv 3p(\frac{p}{3})U_{p-3} \pmod{p^2}$. Thus,

$$H_{[\frac{2p}{3}]}=H_{[\frac{p}{3}]}+\sum_{k=1}^{[2p/3]}\frac{(-1)^{k-1}}{k}\equiv-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2+2p\Big(\frac{p}{3}\Big)U_{p-3} \pmod{p^2}.$$

By [14, Theorem 3.3], $H^{(2)}_{[\frac{p}{3}]} \equiv 3\left(\frac{p}{3}\right)U_{p-3} \pmod p$. To complete the proof, we note that $\sum_{k=0}^{p-1}\frac{1}{k^2} \equiv 0 \pmod p$ (see [12, Theorem 5.1]) and so

$$H^{(2)}_{[\frac{2p}{3}]} = \sum_{k=1}^{p-1}\frac{1}{k^2} - \sum_{k=1}^{[p/3]}\frac{1}{(p-k)^2} \equiv -H^{(2)}_{[\frac{p}{3}]} \pmod p.$$

Theorem 3.3. *Let p be a prime with $p>3$. For $p\equiv 1 \pmod 3$ and so $p=x^2+3y^2$ we have*

$$\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k} \equiv 4x^2-2p \pmod{p^3}.$$

For $p\equiv 2 \pmod 3$ we have

$$\begin{aligned}&\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k}\\ &\equiv -2\binom{(p-1)/2}{(p-5)/6}^2\Big(1+p\Big(2+\frac{4}{3}q_p(2)-\frac{3}{2}q_p(3)\Big)+p^2\Big(1+\frac{8}{3}q_p(2)+\frac{2}{9}q_p(2)^2\\ &\quad -3q_p(3)-2q_p(2)q_p(3)+\frac{15}{8}q_p(3)^2+\frac{3}{4}U_{p-3}\Big)\Big)-\frac{1}{2}p^2\binom{(p-1)/2}{(p-5)/6}^{-2} \pmod{p^3}.\end{aligned}$$

Proof. For $p=x^2+3y^2\equiv 1 \pmod 3$ we have $\langle\frac{2}{3}\rangle_p=\frac{p+2}{3}\equiv 1 \pmod 2$ and $\frac{2}{3}-\langle\frac{2}{3}\rangle_p=-\frac{p}{3}$. Thus, from Theorem 2.1 and the well known fact $\binom{(p-1)/2}{(p-1)/6}\equiv 2x\left(\frac{x}{3}\right) \pmod p$ (see [2,p.283]) we obtain

$$S_p\Big(\frac{2}{3}\Big) \equiv \frac{-\frac{2}{9}p^2}{\frac{4}{9}\binom{(p-1)/2}{(p-1)/6}^2} \equiv -\frac{p^2}{8x^2} \pmod{p^3}.$$

Taking $a=-\frac{1}{3}$ in Theorem 3.1 and then applying the above and (1.2) yields

$$\begin{aligned}\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k} &\equiv S_p\Big(-\frac{1}{3}\Big)-2S_p\Big(\frac{2}{3}\Big)\equiv 4x^2-2p-\frac{p^2}{4x^2}-2\Big(-\frac{p^2}{8x^2}\Big)\\ &= 4x^2-2p \pmod{p^3}.\end{aligned}$$

Now assume that $p\equiv 2 \pmod 3$. By Lemma 3.3,

$$\begin{aligned}\frac{2}{3}H_{\frac{2(p+1)}{3}}+\frac{1}{3}H_{\frac{p+1}{3}} &\equiv \frac{2}{3}\Big(\frac{1}{2(p+1)/3}-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2+2p\Big(\frac{p}{3}\Big)U_{p-3}\Big)\\ &\quad+\frac{1}{3}\Big(\frac{1}{(p+1)/3}-\frac{3}{2}q_p(3)+\frac{3}{4}pq_p(3)^2-p\Big(\frac{p}{3}\Big)U_{p-3}\Big)\\ &\equiv 2-\frac{3}{2}q_p(3)+p\Big(-2+\frac{3}{4}q_p(3)^2-U_{p-3}\Big) \pmod{p^2}.\end{aligned}$$

From Lemma 3.3 we also have

$$H^{(2)}_{\frac{p+1}{3}} = \frac{9}{(p+1)^2}+H^{(2)}_{[\frac{p}{3}]} \equiv 9-3U_{p-3} \pmod p,$$

$$H^{(2)}_{\frac{2(p+1)}{3}} = \frac{9}{4(p+1)^2} + H^{(2)}_{[\frac{2p}{3}]} \equiv \frac{9}{4} + 3U_{p-3} \pmod p.$$

Set $a = -\frac{1}{3}$. Then $\langle a\rangle_p = \frac{2p-1}{3}$ and $a' = -\frac{2}{3}$. From Theorem 3.1 and the above,

$$\begin{aligned}
&\sum_{k=0}^{p-2} \frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k} \\
&\equiv -2\binom{(p-1)/2}{(p+1)/3}^2 \Big(1 + p(\frac{2}{3}H_{\frac{2(p+1)}{3}} + \frac{1}{3}H_{\frac{p+1}{3}} + \frac{4}{3}q_p(2)) \\
&\quad + \frac{p^2}{2}\Big(-\frac{4}{3}q_p(2)^2 + (\frac{2}{3}H_{\frac{2(p+1)}{3}} + \frac{1}{3}H_{\frac{p+1}{3}} + \frac{4}{3}q_p(2))^2 - \frac{1}{18}H^{(2)}_{\frac{p+1}{3}} + \frac{10}{9}H^{(2)}_{\frac{2(p+1)}{3}}\Big)\Big) \\
&\quad - \frac{1}{2}p^2\binom{(p-1)/2}{(p+1)/3}^{-2} \\
&\equiv -2\binom{(p-1)/2}{(p-5)/6}^2 \Big(1 + p\Big(2 + \frac{4}{3}q_p(2) - \frac{3}{2}q_p(3)\Big) + p^2\Big(-2 + \frac{3}{4}q_p(3)^2 - U_{p-3}\Big) \\
&\quad + \frac{p^2}{2}\Big(-\frac{4}{3}q_p(2)^2 + \big(2 + \frac{4}{3}q_p(2) - \frac{3}{2}q_p(3)\big)^2 - \frac{1}{18}(9 - 3U_{p-3}) + \frac{10}{9}\big(\frac{9}{4} + 3U_{p-3}\big)\Big)\Big) \\
&\quad - \frac{1}{2}p^2\binom{(p-1)/2}{(p-5)/6}^{-2} \\
&\equiv -2\binom{(p-1)/2}{(p-5)/6}^2 \Big(1 + p\Big(2 + \frac{4}{3}q_p(2) - \frac{3}{2}q_p(3)\Big) + p^2\Big(1 + \frac{8}{3}q_p(2) + \frac{2}{9}q_p(2)^2 \\
&\quad - 3q_p(3) - 2q_p(2)q_p(3) + \frac{15}{8}q_p(3)^2 + \frac{3}{4}U_{p-3}\Big)\Big) - \frac{1}{2}p^2\binom{(p-1)/2}{(p-5)/6}^{-2} \pmod{p^3}.
\end{aligned}$$

This completes the proof.

Theorem 3.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-2} \frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^3} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod 8, \\ -\frac{1}{3}R_2(p) \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

Proof. Taking $a = -\frac{1}{4}, -\frac{3}{4}$ in Theorem 3.1 gives

$$\sum_{k=0}^{p-2} \frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k} \equiv S_p\Big(-\frac{1}{4}\Big) - 3S_p\Big(\frac{3}{4}\Big) \equiv S_p\Big(-\frac{3}{4}\Big) - \frac{1}{3}S_p\Big(\frac{1}{4}\Big) \pmod{p^3}.$$

For $p = x^2 + 2y^2 \equiv 1 \pmod 8$ we see that $\langle \frac{3}{4}\rangle_p = \frac{p+3}{4} \equiv 1 \pmod 2$ and $\frac{3}{4} - \langle \frac{3}{4}\rangle_p = -\frac{p}{4}$. By Theorem 2.1 and the fact that $\binom{(p-1)/2}{(p-1)/8} \equiv 2(-1)^{\frac{p-1}{8} + \frac{x-1}{2}} x \pmod p$ (see [2,p.272]),

$$S_p\Big(\frac{3}{4}\Big) \equiv \frac{-\frac{1}{4}(1 - \frac{1}{4})p^2}{(\frac{3}{4})^2\binom{(p-1)/2}{(p-1)/8}^2} \equiv -\frac{p^2}{12x^2} \pmod{p^3}.$$

Recall that $S_p(-\frac{1}{4}) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}$ by (1.3). We then get

$$\sum_{k=0}^{p-2} \frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k} \equiv S_p\Big(-\frac{1}{4}\Big) - 3S_p\Big(\frac{3}{4}\Big) \equiv 4x^2 - 2p - \frac{p^2}{4x^2} - 3\Big(-\frac{p^2}{12x^2}\Big)$$

$$= 4x^2 - 2p \pmod{p^3}.$$

For $p = x^2 + 2y^2 \equiv 3 \pmod 8$ we see that $\langle \frac{1}{4}\rangle_p = \frac{p+1}{4} \equiv 1 \pmod 2$ and $\frac{1}{4} - \langle \frac{1}{4}\rangle_p = -\frac{p}{4}$. By Theorem 2.1 and the fact that $\binom{(p-1)/2}{(p-3)/8} \equiv 2(-1)^{\frac{p+5}{8}+\frac{x-1}{2}} x \pmod p$ (see [2,p.417]), we deduce that

$$S_p\Big(\frac{1}{4}\Big) \equiv \frac{-\frac{1}{4}(1-\frac{1}{4})p^2}{(\frac{1}{4})^2\binom{(p-1)/2}{(p-3)/8}^2} \equiv -\frac{3p^2}{4x^2} \pmod{p^3}.$$

Recall that $S_p(-\frac{3}{4}) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}$. We then get

$$\begin{aligned}\sum_{k=0}^{p-2} \frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k} &\equiv S_p\Big(-\frac{3}{4}\Big) - \frac{1}{3}S_p\Big(\frac{1}{4}\Big) \equiv 4x^2 - 2p - \frac{p^2}{4x^2} - \frac{1}{3}\Big(-\frac{3p^2}{4x^2}\Big)\\ &= 4x^2 - 2p \pmod{p^3}.\end{aligned}$$

For $p \equiv 5 \pmod 8$, taking $a = -\frac{1}{4}$, $\langle a\rangle_p = \frac{p-1}{4}$ and $a' = -\frac{1}{4}$ in Theorem 3.1 and then applying the fact that $H_{[\frac{p}{4}]} \equiv -3q_p(2) \pmod p$ (see [23,(2.4)]) yields

$$\begin{aligned}&\sum_{k=0}^{p-2} \frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k}\\ &\equiv -3\binom{(p-1)/2}{(p+3)/8}^2\Big(1 + p\big(\frac{3}{2}H_{\frac{p+3}{4}} - \frac{1}{2}H_{\frac{p+3}{8}} + \frac{1}{2}q_p(2)\big)\Big)\\ &\equiv -3\Big(\frac{3p+1}{p+3}\Big)^2\binom{(p-1)/2}{(p-5)/8}^2\Big(1 + p\big(\frac{3}{2}(\frac{4}{p+3} - 3q_p(2)) - \frac{1}{2}(\frac{8}{p+3} + H_{[\frac{p}{8}]}) + \frac{1}{2}q_p(2)\big)\Big)\\ &\equiv \Big(-\frac{1}{3} - \frac{16}{9}p\Big)\binom{(p-1)/2}{(p-5)/8}^2\Big(1 + p\Big(\frac{2}{3} - 4q_p(2) - \frac{1}{2}H_{[\frac{p}{8}]}\Big)\Big) \equiv -\frac{1}{3}R_2(p) \pmod{p^2}.\end{aligned}$$

For $p \equiv 7 \pmod 8$, taking $a = -\frac{3}{4}$, $\langle a\rangle_p = \frac{p-3}{4}$ and $a' = -\frac{1}{4}$ in Theorem 3.1 and then applying the fact that $H_{[\frac{p}{4}]} \equiv -3q_p(2) \pmod p$ yields

$$\begin{aligned}&\sum_{k=0}^{p-2} \frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k}\\ &\equiv -\frac{1}{3}\binom{(p-1)/2}{(p+1)/8}^2\Big(1 + p\big(\frac{3}{2}H_{\frac{p+1}{4}} - \frac{1}{2}H_{\frac{p+1}{8}} + \frac{1}{2}q_p(2)\big)\Big)\\ &\equiv -\frac{1}{3}\cdot 9\binom{(p-1)/2}{(p-7)/8}^2\Big(1 + p\big(\frac{3}{2}(\frac{4}{p+1} - 3q_p(2)) - \frac{1}{2}(\frac{8}{p+1} + H_{[\frac{p}{8}]}) + \frac{1}{2}q_p(2)\big)\Big)\\ &\equiv -\frac{1}{3}\cdot 9\binom{(p-1)/2}{(p-7)/8}^2\Big(1 + p\big(2 - 4q_p(2) - \frac{1}{2}H_{[\frac{p}{8}]}\big)\Big) = -\frac{1}{3}R_2(p) \pmod{p^2}.\end{aligned}$$

Putting all the above together proves the theorem.

Theorem 3.5. *Let p be a prime with $p > 3$. Then*

$$\sum_{k=0}^{p-2} \frac{\binom{3k}{k}\binom{6k}{3k}C_k}{1728^k}$$

$$\equiv \begin{cases} (\frac{p}{3})(4x^2-2p) \pmod{p^2} & \textit{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{1}{5}\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\Big(1+p\big(10-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\Big) \pmod{p^2} & \textit{if } 12\mid p-7,\\ -5\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\Big(1+p\big(2-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\Big) \pmod{p^2} & \textit{if } 12\mid p-11. \end{cases}$$

Proof. For $p=x^2+4y^2\equiv 1 \pmod 4$, taking $a=-\frac{1}{6}$ in Theorem 3.1 and then applying Theorem 2.1 yields

$$\sum_{k=0}^{p-2}\frac{\binom{3k}{k}\binom{6k}{3k}C_k}{1728^k}\equiv S_p\Big(-\frac{1}{6}\Big)-5S_p\Big(\frac{5}{6}\Big)\equiv S_p\Big(-\frac{1}{6}\Big)\equiv\Big(\frac{p}{3}\Big)(4x^2-2p) \pmod{p^2}.$$

For $p\equiv 7 \pmod{12}$, taking $a=-\frac{1}{6}$, $\langle a\rangle_p=\frac{p-1}{6}$ and $a'=-\frac{1}{6}$ in Theorem 3.1 and then applying the fact that $H_{[\frac{p}{6}]}\equiv -2q_p(2)-\frac{3}{2}q_p(3) \pmod p$ (see [23,(2.4)]) yields

$$\begin{aligned}\sum_{k=0}^{p-2}\frac{\binom{3k}{k}\binom{6k}{3k}C_k}{1728^k}&\equiv -5\binom{(p-1)/2}{(p+5)/12}^2\Big(1+p\Big(\frac{5}{3}H_{\frac{p+5}{6}}-\frac{2}{3}H_{\frac{p+5}{12}}+\frac{1}{3}q_p(2)\Big)\Big)\\ &\equiv -\frac{1}{5}\binom{(p-1)/2}{[p/12]}^2\Big(1+\frac{48}{5}p\Big)\\ &\quad\times\Big(1+p\Big(\frac{5}{3}\Big(\frac{6}{p+5}-2q_p(2)-\frac{3}{2}q_p(3)\Big)-\frac{2}{3}\Big(\frac{12}{p+5}+H_{[\frac{p}{12}]}\Big)+\frac{1}{3}q_p(2)\Big)\Big)\\ &\equiv -\frac{1}{5}\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\Big(1+p\big(10-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\Big) \pmod{p^2}.\end{aligned}$$

For $p\equiv 11 \pmod{12}$, taking $a=-\frac{5}{6}$, $\langle a\rangle_p=\frac{p-5}{6}$ and $a'=-\frac{1}{6}$ in Theorem 3.1 and then applying the fact that $H_{[\frac{p}{6}]}\equiv -2q_p(2)-\frac{3}{2}q_p(3) \pmod p$ yields

$$\begin{aligned}&\sum_{k=0}^{p-2}\frac{\binom{3k}{k}\binom{6k}{3k}C_k}{1728^k}\\ &\equiv -\frac{1}{5}\binom{(p-1)/2}{(p+1)/12}^2\Big(1+p\Big(\frac{5}{3}H_{\frac{p+1}{6}}-\frac{2}{3}H_{\frac{p+1}{12}}+\frac{1}{3}q_p(2)\Big)\Big)\\ &\equiv -5\binom{(p-1)/2}{[p/12]}^2\Big(1+p\Big(\frac{5}{3}\Big(\frac{6}{p+1}-2q_p(2)-\frac{3}{2}q_p(3)\Big)-\frac{2}{3}\Big(\frac{12}{p+1}+H_{[\frac{p}{12}]}\Big)+\frac{1}{3}q_p(2)\Big)\Big)\\ &\equiv -5\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\Big(1+p\big(2-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\Big) \pmod{p^2}.\end{aligned}$$

This completes the proof.

Remark 3.1 Since $C_k=\binom{2k}{k}-\binom{2k}{k+1}$ for $k\ge 0$, from [26] one may deduce that for any prime $p>3$,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2C_k}{64^k}\equiv 4x^2-2p \pmod{p^2}\quad\text{for}\quad p=x^2+4y^2\equiv 1 \pmod 4,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k}\equiv 4x^2-2p \pmod{p^2}\quad\text{for}\quad p=x^2+3y^2\equiv 1 \pmod 3,$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k}\equiv 4x^2-2p \pmod{p^2}\quad\text{for}\quad p=x^2+2y^2\equiv 1,3 \pmod 8,$$
$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}C_k}{1728^k}\equiv 4x^2-2p \pmod{p^2}\quad\text{for}\quad p=x^2+4y^2\equiv 1 \pmod 4.$$

In [25], the author made a conjecture equivalent to

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-2}\frac{\binom{3k}{k}\binom{6k}{3k}C_k}{1728^k}\equiv\begin{cases}4x^2-2p \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{3}{5}R_1(p) \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

For the conjectures concerning Theorems 3.2-3.4 see [25, Conjectures 5.4, 5.11 and 5.16].

4. The congruence for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{C_{k-1}}{4^k}$ modulo p^3

Lemma 4.1. *For any positive integer n and real number a we have*

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k(2k-1)}\\&=-(2a^2+2a+1)S_n(a)-2(a+1)^2S_n(a+1)\\&\quad+\frac{(2a+1)(2a+2)n}{2n-1}\binom{a}{n-1}\binom{-2-a}{n-1}\binom{2n-1}{n-1}\frac{1}{4^{n-1}}.\end{aligned}$$

Proof. Put

$$\begin{aligned}F(a,k)&=\Big(\binom{a}{k}\binom{-1-a}{k}\big(\frac{1}{2k-1}+(2a^2+2a+1)\big)\\&\quad+2(a+1)^2\binom{a+1}{k}\binom{-2-a}{k}\Big)\binom{2k}{k}\frac{1}{4^k},\\ G(a,k)&=\frac{(2a+1)(2a+2)k}{2k-1}\binom{a}{k-1}\binom{-2-a}{k-1}\binom{2k-1}{k-1}\frac{1}{4^{k-1}}.\end{aligned}$$

It is easy to check that $F(a,k)=G(a,k+1)-G(a,k)$. Thus,

$$\sum_{k=0}^{n-1}F(a,k)=\sum_{k=0}^{n-1}(G(a,k+1)-G(a,k))=G(a,n)-G(a,0)=G(a,n).$$

This yields the result.

Theorem 4.1. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0,-1 \pmod p$ and $a'=(a-\langle a\rangle_p)/p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{C_{k-1}}{4^k}$$

$$\equiv -\big(a^2+a+\frac{1}{2}\big)S_p(a)-(a+1)^2S_p(a+1)-(2a+1)\frac{a'(a'+1)}{a+1}p^3 \pmod{p^4}.$$

Proof. Since $\binom{2k}{k}\frac{1}{2k-1}=2C_{k-1}$ for $k\ge 0$, putting $n=p$ in Lemma 4.1 and then applying (3.1) yields the result.

Corollary 4.1. *Let p be an odd prime, $a\in\mathbb{Z}_p$ and $a\not\equiv 0,-1 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\frac{C_{k-1}}{4^k}\equiv -\frac{1}{2}S_p(a)-a(a+1)\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\frac{C_k}{4^k} \pmod{p^3}.$$

Proof. Combining Theorem 3.1 with Theorem 4.1 yields the result.

Theorem 4.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}C_{k-1}^2}{64^k}$$
$$\equiv\begin{cases}\frac{x^2}{2}-\frac{p}{4}-\frac{p^2}{8x^2} \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ \frac{3(p+1)^2}{2^{p+2}}\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^2+\frac{3}{16}p^2\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^2E_{p-3}-\frac{p^2}{8}\binom{\frac{p-1}{2}}{\frac{p-3}{4}}^{-2} \pmod{p^3} & \text{if } 4\mid p-3.\end{cases}$$

Proof. Taking $a=\frac12$ in Theorem 4.1 and then applying Lemma 2.1 gives

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{\frac12}{k}\binom{-\frac32}{k}\frac{C_{k-1}}{4^k}&\equiv -\Big(\frac14+\frac12+\frac12\Big)S_p\Big(\frac12\Big)-\Big(\frac12+1\Big)^2S_p\Big(\frac12+1\Big)\\&\equiv -\frac54 S_p\Big(\frac12\Big)-\frac14 S_p\Big(-\frac12\Big) \pmod{p^3}.\end{aligned}$$

Since $C_{k-1}=\binom{2k}{k}\frac{1}{2(2k-1)}$ and $\binom{1/2}{k}\binom{-3/2}{k}=-(1+\frac{2}{2k-1})\binom{2k}{k}^2\frac{1}{16^k}$ by (2.6), from (2.7) and the above we deduce that

$$\begin{aligned}4\sum_{k=0}^{p-1}\frac{\binom{2k}{k}C_{k-1}^2}{64^k}&=-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2C_{k-1}}{64^k}-\sum_{k=0}^{p-1}\binom{\frac12}{k}\binom{-\frac32}{k}\frac{C_{k-1}}{4^k}\\&\equiv\frac14\Big(S_p\Big(\frac12\Big)+S_p\Big(-\frac12\Big)\Big)+\frac54S_p\Big(\frac12\Big)+\frac14S_p\Big(-\frac12\Big)\\&=\frac32S_p\Big(\frac12\Big)+\frac12S_p\Big(-\frac12\Big) \pmod{p^3}.\end{aligned}$$

Now, applying (1.1) and Lemma 2.5 yields the result.

Theorem 4.3. *Let $p>3$ be a prime. For $p\equiv 1 \pmod 3$ and so $p=x^2+3y^2$ we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}C_{k-1}}{108^k}\equiv -\frac{10}{9}x^2+\frac59p+\frac{p^2}{8x^2} \pmod{p^3}.$$

For $p\equiv 2 \pmod 3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}C_{k-1}}{108^k}$$

$$\equiv -\frac{4}{9}\binom{\frac{p-1}{2}}{\frac{p-5}{6}}^2\Big(1+p\Big(2+\frac{4}{3}q_p(2)-\frac{3}{2}q_p(3)\Big)+p^2\Big(1+\frac{8}{3}q_p(2)+\frac{2}{9}q_p(2)^2$$
$$-3q_p(3)-2q_p(2)q_p(3)+\frac{15}{8}q_p(3)^2+\frac{3}{4}U_{p-3}\Big)\Big)+\frac{5}{36}p^2\binom{\frac{p-1}{2}}{\frac{p-5}{6}}^{-2}\pmod{p^3}.$$

Proof. Taking $a=-\frac{1}{3}$ in Corollary 4.1 gives

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}C_{k-1}}{108^k}\equiv -\frac{1}{2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}+\frac{2}{9}\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k}\pmod{p^3}.$$

Now applying (1.2) and Theorem 3.3 yields the result.

Theorem 4.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}C_{k-1}}{256^k}\equiv\begin{cases}-\frac{5}{4}x^2+\frac{5}{8}p+\frac{p^2}{8x^2}\pmod{p^3} & \text{if } p=x^2+2y^2\equiv 1,3\pmod 8,\\ -\frac{1}{16}R_2(p)\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Proof. Taking $a=-\frac{1}{4}$ in Corollary 4.1 gives

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}C_{k-1}}{256^k}\equiv -\frac{1}{2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}+\frac{3}{16}\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k}\pmod{p^3}.$$

Now applying (1.3) and Theorem 3.4 yields the result.

Theorem 4.5. *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}C_{k-1}}{12^{3k}}$$
$$\equiv\begin{cases}(\frac{p}{3})(-\frac{13}{9}x^2+\frac{13}{18}p)\pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1\pmod 4,\\ -\frac{1}{36}\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\Big(1+p\big(10-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\Big)\pmod{p^2} & \text{if } 12\mid p-7,\\ -\frac{25}{36}\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\Big(1+p\big(2-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\Big)\pmod{p^2} & \text{if } 12\mid p-11.\end{cases}$$

Proof. Taking $a=-\frac{1}{6}$ in Corollary 4.1 gives

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}C_{k-1}}{12^{3k}}\equiv -\frac{1}{2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}+\frac{5}{36}\sum_{k=0}^{p-2}\frac{\binom{3k}{k}\binom{6k}{3k}C_k}{12^{3k}}\pmod{p^3}.$$

Now, applying Theorem 3.5 and the known result for $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}12^{-3k}$ modulo p^2 yields the result.

Remark 4.1 For the conjectures concerning Theorems 4.2-4.5 see [25, Conjectures 5.4, 5.7, 5.11 and 5.16].

5. The congruence for $\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k(k+1)^2}$ modulo p^3

Lemma 5.1. *For any positive integer n and real number $a\neq 0$ we have*

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k(k+1)^2}\\&=2S_n(a)+\frac{2a^2+2a-1}{a^2}S_n(a+1)+\frac{2((2a-1)n+a)}{a^2}\binom{a}{n-1}\binom{-1-a}{n}\binom{2n}{n}\frac{1}{4^n}.\end{aligned}$$

Proof. Let

$$\begin{aligned}F(a,k)&=\left(\binom{a}{k}\binom{-1-a}{k}\Big(\frac{1}{(k+1)^2}-2\Big)-\frac{2a^2+2a-1}{a^2}\binom{a+1}{k}\binom{-2-a}{k}\right)\binom{2k}{k}\frac{1}{4^k},\\G(a,k)&=\frac{2((2a-1)k+a)}{a^2}\binom{a}{k-1}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k}.\end{aligned}$$

It is easy to check that $F(a,k)=G(a,k+1)-G(a,k)$. Thus,

$$\sum_{k=0}^{n-1}F(a,k)=\sum_{k=0}^{n-1}(G(a,k+1)-G(a,k))=G(a,n)-G(a,0)=G(a,n).$$

This yields the result.

Theorem 5.1. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0,-1,-2 \pmod p$ and $a'=(a-\langle a\rangle_p)/p$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k(k+1)^2}\\&\equiv 2S_p(a)+\frac{2a^2+2a-1}{a^2}S_p(a+1)+\frac{4a^3+6a^2-3a+2}{a^3(a+1)(a+2)}a'(a'+1)p^3 \pmod{p^4}.\end{aligned}$$

Proof. Taking $n=p-1$ in Lemma 5.1 and then applying Lemma 3.2 we see that

$$\begin{aligned}&\sum_{k=0}^{p-2}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k(k+1)^2}\\&\equiv 2S_{p-1}(a)+\frac{2a^2+2a-1}{a^2}S_{p-1}(a+1)+\frac{2(a-1)}{a^2(a+2)}\Big(-\frac{a'(a'+1)}{a(a+1)}p^3\Big)\\&\equiv 2S_p(a)+2\frac{a'(a'+1)}{a(a+1)}p^3+\frac{2a^2+2a-1}{a^2}S_p(a+1)\\&\quad+\frac{2a^2+2a-1}{a^2}\cdot\frac{a'(a'+1)}{(a+1)(a+2)}p^3-\frac{2(a-1)a'(a'+1)}{a^3(a+1)(a+2)}p^3\\&=2S_p(a)+\frac{2a^2+2a-1}{a^2}S_p(a+1)+\frac{4a^3+6a^2-3a+2}{a^3(a+1)(a+2)}a'(a'+1)p^3 \pmod{p^4}.\end{aligned}$$

This proves the theorem.

Theorem 5.2. *Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $a\not\equiv 0,-1,-2 \pmod p$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-2}\binom ak\binom{-1-a}k\binom{2k}k\frac 1{4^k(k+1)^2}\\&\equiv\Big(2-\frac 1{a(a+1)}\Big)\sum_{k=0}^{p-2}\binom ak\binom{-1-a}k\frac{C_k}{4^k}+\frac 1{a(a+1)}S_p(a) \pmod{p^3}.\end{aligned}$$

Proof. By Theorems 5.1 and 3.1,

$$\begin{aligned}&\sum_{k=0}^{p-2}\binom ak\binom{-1-a}k\binom{2k}k\frac 1{4^k(k+1)^2}\\&\equiv 2S_p(a)+\frac{2a^2+2a-1}{a^2}S_p(a+1)\\&\equiv 2S_p(a)+\frac{2a^2+2a-1}{a^2}\cdot\frac a{a+1}\Big(\sum_{k=0}^{p-2}\binom ak\binom{-1-a}k\frac{C_k}{4^k}-S_p(a)\Big)\\&=\frac{2a(a+1)-1}{a(a+1)}\sum_{k=0}^{p-2}\binom ak\binom{-1-a}k\frac{C_k}{4^k}+\frac 1{a(a+1)}S_p(a) \pmod{p^3}.\end{aligned}$$

This proves the theorem.

Theorem 5.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{\frac{p-1}2}\frac{\binom{2k}kC_k^2}{64^k}\equiv\begin{cases}8x^2-4p+\frac{p^2}{x^2} \pmod{p^3} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\frac{6(p+1)^2}{2^{p-1}}\binom{(p-1)/2}{(p-3)/4}^2-2p^2\binom{(p-1)/2}{(p-3)/4}^{-2}\\ \qquad -3p^2\binom{(p-1)/2}{(p-3)/4}^2E_{p-3} \pmod{p^3} & \text{if } 4\mid p-3.\end{cases}$$

Proof. Since $p\mid\binom{2k}k$ for $\frac p2<k<p$, taking $a=-\frac12$ in Theorem 5.2 gives

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}kC_k^2}{64^k}\equiv 6\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}k^2C_k}{64^k}-4\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}k^3}{64^k} \pmod{p^3}.$$

Now applying (1.1) and Theorem 3.2 yields the result.

Theorem 5.4. *Let p be a prime with $p>5$. For $p\equiv 1 \pmod 3$ and so $p=x^2+3y^2$ we have*

$$\sum_{k=0}^{p-2}\frac{\binom{3k}kC_k^2}{108^k}\equiv 8x^2-4p+\frac{9p^2}{8x^2} \pmod{p^3}.$$

For $p\equiv 2 \pmod 3$ we have

$$\sum_{k=0}^{p-2}\frac{\binom{3k}kC_k^2}{108^k}$$

$$\equiv -13\binom{(p-1)/2}{(p-5)/6}^2\Big(1+p\Big(2+\frac{4}{3}q_p(2)-\frac{3}{2}q_p(3)\Big)+p^2\Big(1+\frac{8}{3}q_p(2)+\frac{2}{9}q_p(2)^2 - 3q_p(3)-2q_p(2)q_p(3)+\frac{15}{8}q_p(3)^2+\frac{3}{4}U_{p-3}\Big)\Big)-p^2\binom{(p-1)/2}{(p-5)/6}^{-2} \pmod{p^3}.$$

Proof. Taking $a=-\frac{1}{3}$ in Theorem 5.2 yields

$$\sum_{k=0}^{p-2}\frac{\binom{3k}{k}C_k^2}{108^k}\equiv \frac{13}{2}\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}C_k}{108^k}-\frac{9}{2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \pmod{p^3}.$$

Now applying (1.2) and Theorem 3.3 yields the result.

Theorem 5.5. *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{p-2}\frac{\binom{4k}{2k}C_k^2}{256^k}\equiv\begin{cases}8x^2-4p+\dfrac{4p^2}{3x^2} \pmod{p^3} & \text{if } p=x^2+2y^2\equiv 1,3 \pmod 8,\\ -\dfrac{22}{9}R_2(p) \pmod{p^2} & \text{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Proof. Taking $a=-\frac{1}{4}$ in Theorem 5.2 yields

$$\sum_{k=0}^{p-2}\frac{\binom{4k}{2k}C_k^2}{256^k}\equiv \frac{22}{3}\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{4k}{2k}C_k}{256^k}-\frac{16}{3}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \pmod{p^3}.$$

Now applying (1.3) and Theorem 3.4 yields the result.

Theorem 5.6. *Let p be a prime with $p\neq 2,3,11$. Then*

$$\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k(k+1)^2}\equiv\begin{cases}(\frac{p}{3})(8x^2-4p) \pmod{p^2} & \text{if } p=x^2+4y^2\equiv 1 \pmod 4,\\ -\dfrac{46}{25}\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\big(1+p\big(10-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\big) \pmod{p^2} & \text{if } 12\mid p-7,\\ -46\binom{\frac{p-1}{2}}{[\frac{p}{12}]}^2\big(1+p\big(2-3q_p(2)-\frac{5}{2}q_p(3)-\frac{2}{3}H_{[\frac{p}{12}]}\big)\big) \pmod{p^2} & \text{if } 12\mid p-11.\end{cases}$$

Proof. Taking $a=-\frac{1}{6}$ in Theorem 5.2 yields

$$\sum_{k=0}^{p-2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k(k+1)^2}\equiv \frac{46}{5}\sum_{k=0}^{p-2}\frac{\binom{3k}{k}\binom{6k}{3k}C_k}{1728^k}-\frac{36}{5}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k} \pmod{p^3}.$$

Now applying Theorem 3.5 and the known result for $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}1728^{-k}$ modulo p^2 yields the result.

Remark 5.1 Let $p>5$ be a prime. The congruence for $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}C_k^2 64^{-k}$ modulo p^2 can be deduced from [29,(4)]. In [25] the author conjectured the congruences modulo p^2 for the sums in Theorem 5.4-5.6.

Calculations by Maple suggest the following conjecture.
Conjecture 5.1. *Let p be an odd prime.*
(i) *For $p \equiv 1,2,4 \pmod 7$ and so $p = x^2 + 7y^2$ we have*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k} C_k^2 \equiv -68y^2 + p - \frac{p^2}{4y^2} \pmod{p^3},$$

$$(-1)^{\frac{p-1}{2}} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k^2}{4096^k} \equiv -1136y^2 + 64p + \frac{2p^2}{y^2} \pmod{p^3}.$$

(ii) *For $p \equiv 1 \pmod 3$ and so $p = x^2 + 3y^2$ we have*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k^2}{16^k} \equiv -24y^2 + 2p - \frac{p^2}{2y^2} \pmod{p^3},$$

$$(-1)^{\frac{p-1}{2}} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k^2}{256^k} \equiv -48y^2 + 8p \pmod{p^3}.$$

(iii) *For $p \equiv 1 \pmod 4$ and so $p = x^2 + 4y^2$ we have*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k^2}{(-8)^k} \equiv -32y^2 + p - \frac{7p^2}{16y^2} \pmod{p^3},$$

$$(-1)^{\frac{p-1}{4}} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k^2}{(-512)^k} \equiv 64y^2 - 8p - \frac{p^2}{y^2} \pmod{p^3}.$$

(iv) *For $p \equiv 1,3 \pmod 8$ and so $p = x^2 + 2y^2$ we have*

$$(-1)^{\frac{p-1}{2}} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} C_k^2}{(-64)^k} \equiv -8y^2 - \frac{p^2}{y^2} \pmod{p^3}.$$

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