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Some further properties of even and odd sequences

Zhi-Hong Sun

School of Mathematical Sciences
Huaiyin Normal University,
Huaian, Jiangsu 223300, P.R. China
zhsun@hytc.edu.cn

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Abstract

In this paper we continue to investigate the properties of those sequences $\{a_n\}$ satisfying the condition $\sum_{k=0}^n \binom nk (-1)^k a_k = \pm a_n$ $(n \ge 0)$. As applications we deduce some recurrence relations and congruences for Bernoulli and Euler numbers.

Keywords: even sequence; odd sequence; congruence; Bernoulli number; Euler number; Fibonacci number.

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1. Introduction

The classical binomial inversion formula states that $a_n = \sum_{k=0}^n \binom nk (-1)^k b_k$ $(n = 0, 1, 2, \ldots)$ if and only if $b_n = \sum_{k=0}^n \binom nk (-1)^k a_k (n = 0, 1, 2, \ldots)$. Following [10] we continue to study those sequences $\{a_n\}$ with the property $\sum_{k=0}^n \binom nk (-1)^k a_k = \pm a_n$ $(n = 0, 1, 2, \ldots)$.

Definition 1.1. *If a sequence $\{a_n\}$ satisfies the relation*

$$\sum_{k=0}^n \binom nk (-1)^k a_k = a_n \quad (n = 0, 1, 2, \ldots),$$

we say that $\{a_n\}$ is an even sequence. If $\{a_n\}$ satisfies the relation

$$\sum_{k=0}^n \binom nk (-1)^k a_k = -a_n \quad (n = 0, 1, 2, \ldots),$$

we say that $\{a_n\}$ is an odd sequence.

From [10, Theorem 3.2] we know that $\{a_n\}$ is an even (odd) sequence if and only if $\mathrm{e}^{-x/2}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ is an even (odd) function. Throughout this paper, S^+ denotes the set of even sequences, and S^- denotes the set of odd sequences. In [10] the author stated that

$$\Big\{\frac{1}{2^n}\Big\},\ \Big\{\binom{n+2m-1}{m}^{-1}\Big\},\ \Big\{\binom{2n}{n}2^{-2n}\Big\},\ \Big\{(-1)^n\int_0^{-1}\binom{x}{n}dx\Big\} \in S^+.$$

Let $\{B_n\}$ be the Bernoulli numbers given by $B_0 = 1$ and $\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0$ $(n \ge 2)$. It is well known that $B_1 = -\frac{1}{2}$ and $B_{2m+1} = 0$ for $m \ge 1$. Thus,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k \cdot (-1)^k B_k = B_n + \sum_{k=0}^{n-1}\binom{n}{k}B_k = (-1)^n B_n$$

and so $\{(-1)^n B_n\} \in S^+$ as claimed in [10]. It is also known that $\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{\mathrm{e}^x - 1}$ $(|x| < 2\pi)$. Thus, for $|x| < 2\pi$,

$$\sum_{n=0}^{\infty}(-1)^n(2^n-1)B_n\frac{x^n}{n!} = \frac{-2x}{\mathrm{e}^{-2x}-1} - \frac{-x}{\mathrm{e}^{-x}-1} = \frac{x}{\mathrm{e}^{-x}+1}.$$

Since $\mathrm{e}^{-x/2}x/(\mathrm{e}^{-x}+1)$ is an odd function, we deduce that $\{(-1)^n(2^n-1)B_n\} \in S^-$.

The Euler numbers $\{E_n\}$ is defined by $\frac{2\mathrm{e}^t}{\mathrm{e}^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$ $(|t| < \frac{\pi}{2})$, which is equivalent to (see [4]) $E_0 = 1$, $E_{2n-1} = 0$ and $\sum_{r=0}^{n}\binom{2n}{2r}E_{2r} = 0$ $(n \ge 1)$. It is clear that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{E_n - 1}{2^n}\cdot\frac{t^n}{n!} &= \sum_{n=0}^{\infty}E_n\frac{(t/2)^n}{n!} - \sum_{n=0}^{\infty}\frac{(t/2)^n}{n!}\\ &= \frac{2\mathrm{e}^{\frac{t}{2}}}{\mathrm{e}^t+1} - \mathrm{e}^{\frac{t}{2}} = \mathrm{e}^{\frac{t}{2}}\cdot\frac{1-\mathrm{e}^t}{1+\mathrm{e}^t} \quad \big(|t| < \pi\big).\end{aligned}$$

As $\frac{1-\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}} = \frac{\mathrm{e}^t-1}{\mathrm{e}^t+1}$, we see that $\mathrm{e}^{-\frac{t}{2}}\sum_{n=0}^{\infty}\frac{E_n-1}{2^n}\cdot\frac{t^n}{n!}$ is an odd function. Thus $\big\{\frac{E_n-1}{2^n}\big\}$ is an odd sequence.

For two numbers b and c, let $\{U_n(b,c)\}$ and $\{V_n(b,c)\}$ be the Lucas sequences given by

$$U_0(b,c) = 0,\ U_1(b,c) = 1,\ U_{n+1}(b,c) = bU_n(b,c) - cU_{n-1}(b,c)\ (n \ge 1)$$

and

$$V_0(b,c) = 2,\ V_1(b,c) = b,\ V_{n+1}(b,c) = bV_n(b,c) - cV_{n-1}(b,c)\ (n \ge 1).$$

It is well known that (see [14])

$$U_n(b,c) = \begin{cases}\frac{1}{\sqrt{b^2-4c}}\Big\{\Big(\frac{b+\sqrt{b^2-4c}}{2}\Big)^n - \Big(\frac{b-\sqrt{b^2-4c}}{2}\Big)^n\Big\} & \text{if } b^2-4c \ne 0,\\ n\big(\frac{b}{2}\big)^{n-1} & \text{if } b^2-4c = 0\end{cases}$$

and

$$V_n(b,c) = \Big(\frac{b+\sqrt{b^2-4c}}{2}\Big)^n + \Big(\frac{b-\sqrt{b^2-4c}}{2}\Big)^n.$$

From this one can easily see that for $b \ne 0$, $\{U_n(b,c)/b^n\}$ is an odd sequence and $\{V_n(b,c)/b^n\}$ is an even sequence. We note that $F_n = U_n(1,-1)$ is the Fibonacci sequence and $n = U_n(2,1)$.

Let $\{A_n\}$ be an even sequence or an odd sequence. In Section 2 we deduce new recurrence formulas for $\{A_n\}$ and give a criterion for polynomials $P_m(x)$ with the property $P_m(1-x)=(-1)^mP_m(x)$, in Section 3 we establish a transformation formula for $\sum_{k=0}^n\binom nk A_k$, in Section 4 we give congruences for $\sum_{k=1}^{p-1}\frac{A_{k+1}}{k}$, $\sum_{k=1}^{p-1}\frac{A_k}{k}$ and $\sum_{k=0}^{p-2}\frac{A_k}{k+1}$ modulo p^2, where p is an odd prime. As applications we establish some recurrence formulas for Bernoulli and Euler numbers. Here are some typical results:

$\star$ If $\{A_n\}$ is an odd sequence, then $\sum_{k=0}^n\binom nk(-1)^kA_{2n-k}=0$. If $\{A_n\}$ is an even sequence, then $\sum_{k=0}^n\binom nk(-1)^k(2n-k)A_{2n-k-1}=0$.

$\star$ If $\{A_k\}$ is an even sequence and n is odd, then

$$\sum_{k=0}^n\binom{\frac n2}{k}(-1)^kA_{n-k}=\sum_{k=0}^n\binom nk\binom{n+k}{k}(-1)^kA_k=0$$

and

$$\sum_{\substack{k=0\\3\mid k}}^n\binom nk A_{n-k}=\frac13\sum_{k=0}^n\binom nk A_k.$$

$\star$ Let m be a positive integer and $P_m(x)=\sum_{k=0}^m a_kx^{m-k}$. Then

$$P_m(1-x)=(-1)^mP_m(x)\iff\sum_{k=0}^n\binom nk\frac{a_k}{\binom mk}=(-1)^n\frac{a_n}{\binom mn}\ (n=0,1,\ldots,m).$$

$\star$ Let p be an odd prime, and let $\{A_k\}$ be an odd sequence of rational p-integers. Then

$$2A_{p+1}-A_p\equiv A_1-p\sum_{k=1}^{p-1}\frac{A_{k+1}}{k}\pmod{p^2}\quad\text{and}\quad\sum_{k=1}^{p-1}\frac{A_k}{p+k}\equiv0\pmod{p^2}.$$

In addition to the above notation, throughout this paper we use the following notation: $[x]$—the greatest integer not exceeding x, $\mathbb N$—the set of positive integers, $\mathbb R$—the set of real numbers, $\mathbb Z_p$—the set of those rational numbers whose denominator is coprime to p, $(\frac ap)$—the Legendre symbol.

2. Recurrence formulas for even and odd sequences

Suppose that $\sum_{k=0}^n\binom nk(-1)^ka_k=\pm a_n$ for $n=0,1,2,\ldots$. Then clearly

$$\sum_{k=0}^n\binom nk(-1)^kka_{k-1}=-n\sum_{r=0}^{n-1}\binom{n-1}{r}(-1)^ra_r=\mp na_{n-1}$$

and

$$\begin{aligned}\sum_{k=0}^n\binom nk(-1)^k\frac{a_{k+1}-a_0/2}{k+1}&=-\frac1{n+1}\sum_{k=0}^n\binom{n+1}{k+1}(-1)^{k+1}(a_{k+1}-a_0/2)\\&=-\frac1{n+1}\Big(\sum_{r=0}^{n+1}\binom{n+1}{r}(-1)^r(a_r-a_0/2)-(a_0-a_0/2)\Big)\end{aligned}$$

$$= -\frac{1}{n+1}\big(\pm a_{n+1} - a_0/2\big) = \mp\frac{a_{n+1} \mp a_0/2}{n+1}.$$

Thus,

$$\{a_n\} \in S^+ \quad \text{implies} \quad \{na_{n-1}\}, \{\frac{a_{n+1} - a_0/2}{n+1}\} \in S^-. \tag{2.1}$$

When $\{a_n\} \in S^-$, we have $a_0 = -a_0$ and so $a_0 = 0$. Therefore, from the above we deduce that

$$\{a_n\} \in S^- \quad \text{implies} \quad \{na_{n-1}\}, \{\frac{a_{n+1}}{n+1}\} \in S^+. \tag{2.2}$$

For $x, y \in \mathbb{R}$ and $n \in \{0, 1, 2, \dots\}$ it is well known that ([2, (3.1)])

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

This is called Vandermonde's identity. Let $a_n = \sum_{k=0}^{n} \binom{n-m}{k}(-1)^{n-k} b_{n-k} \quad (n = 0, 1, 2, \dots)$. Using Vandermonde's identity we see that

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n-m}{k}(-1)^{n-k} a_{n-k} \\
&= \sum_{k=0}^{n} \binom{n-m}{k}(-1)^{n-k} \sum_{j=0}^{n-k} \binom{n-k-m}{j}(-1)^{n-k-j} b_{n-k-j} \\
&= \sum_{s=0}^{n} \binom{n-m}{n-s}(-1)^{s} \sum_{j=0}^{s} \binom{s-m}{j}(-1)^{s-j} b_{s-j} \\
&= \sum_{s=0}^{n} \binom{n-m}{n-s} \sum_{r=0}^{s} \binom{m-r-1}{s-r} b_r = \sum_{r=0}^{n}\sum_{s=r}^{n} \binom{n-m}{n-s}\binom{m-r-1}{s-r} b_r \\
&= \sum_{r=0}^{n} \binom{n-r-1}{n-r} b_r = b_n \qquad (n = 0, 1, 2, \dots).
\end{aligned}$$

Thus,

$$\begin{aligned}
&a_n = \sum_{k=0}^{n} \binom{n-m}{k}(-1)^{n-k} b_{n-k} \quad (n = 0, 1, 2, \dots) \\
&\iff b_n = \sum_{k=0}^{n} \binom{n-m}{k}(-1)^{n-k} a_{n-k} \quad (n = 0, 1, 2, \dots).
\end{aligned} \tag{2.3}$$

Lemma 2.1. *Let $m, p \in \mathbb{R}$ and $\sum_{k=0}^{n} \binom{n-m}{k}(-1)^{n-k} a_{n-k} = \pm a_n$ for $n = 0, 1, 2, \dots$. Then*

$$\sum_{k=0}^{n} \binom{n-p-m}{k}(-1)^{n-k} a_{n-k} = \pm \sum_{k=0}^{n} \binom{p}{k}(-1)^{k} a_{n-k} \quad \textit{for} \quad n = 0, 1, 2, \dots.$$

Proof. Using Vandermonde's identity we see that

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n-p-m}{k}(-1)^{n-k}a_{n-k}\\
&=\sum_{k=0}^{n}\binom{n-p-m}{n-k}(-1)^{k}a_{k}\\
&=\pm\sum_{k=0}^{n}\binom{n-p-m}{n-k}(-1)^{k}\sum_{r=0}^{k}\binom{k-m}{k-r}(-1)^{r}a_{r}\\
&=\pm\sum_{r=0}^{n}\Big\{\sum_{k=r}^{n}\binom{n-p-m}{n-k}(-1)^{k-r}\binom{k-m}{k-r}\Big\}a_{r}\\
&=\pm\sum_{r=0}^{n}\Big\{\sum_{k=r}^{n}\binom{n-p-m}{n-k}\binom{m-1-r}{k-r}\Big\}a_{r}\\
&=\pm\sum_{r=0}^{n}\Big\{\sum_{s=0}^{n-r}\binom{n-p-m}{n-r-s}\binom{m-1-r}{s}\Big\}a_{r}\\
&=\pm\sum_{r=0}^{n}\binom{n-p-r-1}{n-r}a_{r}=\pm\sum_{r=0}^{n}\binom{p}{n-r}(-1)^{n-r}a_{r}\\
&=\pm\sum_{k=0}^{n}\binom{p}{k}(-1)^{k}a_{n-k}.
\end{aligned}$$

So the lemma is proved.

Theorem 2.1. *If $\{A_n\}$ is an odd sequence, then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}A_{2n-k}=0 \quad \textit{for} \quad n=0,1,2,\ldots.$$

If $\{A_n\}$ is an even sequence, for $n=0,1,2,\ldots$ we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(2n-k)A_{2n-k-1}=0 \ \textit{and}\ \sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\frac{A_{2n-k+1}}{2n-k+1}=\frac{(-1)^{n}A_0}{2(2n+1)\binom{2n}{n}}.$$

Moreover, for given even sequence $\{A_k\}$ we also have

$$\sum_{k=0}^{n}\binom{\frac{n}{2}}{k}(-1)^{k}A_{n-k}=0 \quad \textit{for} \quad n=1,3,5,\ldots.$$

Proof. We first assume that $\{A_n\}$ is an odd sequence. Putting $m=0$, $p=n/2$ and $a_n=A_n$ in Lemma 2.1 we see that

$$\sum_{k=0}^{n}\binom{\frac{n}{2}}{k}(-1)^{n-k}A_{n-k}=-\sum_{k=0}^{n}\binom{\frac{n}{2}}{k}(-1)^{k}A_{n-k}.$$

Thus, for even n we have

$$\sum_{k=0}^{n/2}\binom{\frac n2}{k}(-1)^kA_{n-k}=\sum_{k=0}^{n}\binom{\frac n2}{k}(-1)^kA_{n-k}=0.$$

Replacing n with $2n$ we get $\sum_{k=0}^n\binom nk(-1)^kA_{2n-k}=0$ for $n=0,1,2,\ldots$.

Now we assume that $\{A_n\}\in S^+$. By (2.1), $\{nA_{n-1}\}\in S^-$ and $\{\frac{A_{n+1}-A_0/2}{n+1}\}\in S^-$. Applying the above we find that

$$\sum_{k=0}^{n}\binom nk(-1)^k(2n-k)A_{2n-k-1}=\sum_{k=0}^{n}\binom nk(-1)^k\frac{A_{2n-k+1}-A_0/2}{2n-k+1}=0.$$

By [2, (1.40)],

$$\sum_{k=0}^{n}\binom nk(-1)^k\frac{1}{k+z}=\frac{1}{\binom{n+z}{n}z}.$$

Thus,

$$\sum_{k=0}^{n}\binom nk(-1)^k\frac{1}{2n+1-k}=\frac{1}{(2n+1)\binom{n-2n-1}{n}}=\frac{(-1)^n}{(2n+1)\binom{2n}{n}}.$$

Hence

$$\sum_{k=0}^{n}\binom nk(-1)^k\frac{A_{2n-k+1}}{2n-k+1}=\frac{(-1)^nA_0}{2(2n+1)\binom{2n}{n}}.$$

If n is odd, taking $m=0$, $p=n/2$ and $a_n=A_n$ in Lemma 2.1 we deduce that $\sum_{k=0}^n\binom{\frac n2}{k}(-1)^kA_{n-k}=0$. This completes the proof.

Since $\{\frac{E_n-1}{2^n}\}$ is an odd sequence and $E_{2k-1}=0$ for $k\ge1$, from Theorem 2.1 we see that $\sum_{k=0}^n\binom nk(-1)^k\frac{E_{2n-k}-1}{2^{2n-k}}=0$ and so

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}\frac{E_{2n-2k}}{2^{2n-2k}}=\sum_{k=0}^{n}\binom nk(-1)^k\frac{E_{2n-k}}{2^{2n-k}}=\sum_{k=0}^{n}\binom nk\frac{(-1)^k}{2^{2n-k}}=\frac{(-1)^n}{2^{2n}}.$$

That is,

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}2^{2k}E_{2n-2k}=(-1)^n\quad\text{for}\quad n=0,1,2,\ldots. \tag{2.4}$$

Since $\{(-1)^nB_n\}$ is an even sequence, from Theorem 2.1 we have $\sum_{k=0}^n\binom nk(-1)^k(2n-k)(-1)^{2n-k-1}B_{2n-k-1}=0$. As $B_{2m+1}=0$ for $m>1$, we obtain

$$\sum_{r=1}^{[\frac{n+1}{2}]}\binom{n}{2r-1}(2n-2r+1)B_{2n-2r}=0\quad\text{for}\quad n=3,4,5,\ldots. \tag{2.5}$$

From Theorem 2.1 we also have

$$\sum_{k=0}^{n}\binom{n/2}{k}B_{n-k}=0\quad\text{for}\quad n=1,3,5,\ldots. \tag{2.6}$$

Theorem 2.2. *Let $\{a_k\}$ be a given sequence. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(a_k-(-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}a_s\Big)=0\quad\textit{for}\quad n=0,1,2,\ldots.$$

Hence, if $\{A_n\}$ is an even sequence and n is odd, or if $\{A_n\}$ is an odd sequence and n is even, then

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^kA_k=0.$$

Proof. Since $\binom{-x}{k}=(-1)^k\binom{x+k-1}{k}$, using Vandermonde's identity we see that for $m\in\{0,1,\ldots,n\}$,

$$\begin{aligned}&\sum_{k=m}^{n}\binom{n-m}{k-m}(-1)^{n-k}\binom{n+k}{k}\\&=\sum_{k=0}^{n}\binom{n-m}{n-k}(-1)^n\binom{-n-1}{k}=(-1)^n\binom{-m-1}{n}=\binom{m+n}{n}.\end{aligned}$$

Note that $\binom{n}{k}\binom{k}{m}=\binom{n}{m}\binom{n-m}{k-m}$. Applying the above we deduce that

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(a_k-(-1)^{n-k}\sum_{s=0}^{k}\binom{k}{s}a_s\Big)\\&=\sum_{m=0}^{n}a_m\Big(\binom{n}{m}\binom{n+m}{m}-\sum_{k=m}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\binom{k}{m}\Big)\\&=\sum_{m=0}^{n}a_m\Big(\binom{n}{m}\binom{n+m}{m}-\binom{n}{m}\sum_{k=m}^{n}\binom{n-m}{k-m}(-1)^{n-k}\binom{n+k}{k}\Big)\\&=\sum_{m=0}^{n}a_m\cdot 0=0.\end{aligned}$$

Putting $a_k=(-1)^kA_k$ in the above formula we obtain the remaining result.

As an example, taking $A_k=(-1)^kB_k$ in Theorem 2.2 we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}B_k=0\quad\text{for}\quad n=1,3,5,\ldots.\tag{2.7}$$

Let $\{F_n\}$ be the Fibonacci sequence given by $F_0=0$, $F_1=1$ and $F_{n+1}=F_n+F_{n-1}$ ($n\ge 1$). As $\{F_n\}$ is an odd sequence, taking $A_k=F_k$ in Theorem 2.2 we get

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^kF_k=0\quad\text{for}\quad n=0,2,4,\ldots.\tag{2.8}$$

Lemma 2.2. *Suppose that m is a nonnegative integer. Then*

$$\sum_{k=0}^{n}\binom{n-m-1}{k}(-1)^{n-k}a_{n-k}=\pm a_n\ (n=0,1,2,\ldots)$$

if and only if

$$\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{\binom{m}{k}}=\pm(-1)^n\frac{a_n}{\binom{m}{n}}\ (n=0,1,\dots,m)$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k a_{k+m+1}=\pm(-1)^{m+1}a_{n+m+1}\ (n=0,1,2,\dots).$$

Proof. For $n=0,1,\dots,m$ we have $\binom{m}{n}\neq 0$. Set $A_n=(-1)^n\frac{a_n}{\binom{m}{n}}$. As $\binom{n-m-1}{k}\binom{m}{n-k}=(-1)^k\binom{n}{k}\binom{m}{n}$, we see that

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n-m-1}{k}(-1)^{n-k}a_{n-k}\\&=\sum_{k=0}^{n}\binom{n-m-1}{k}\binom{m}{n-k}A_{n-k}=\binom{m}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k A_{n-k}\\&=(-1)^n\binom{m}{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k A_k.\end{aligned}$$

Thus,

$$\sum_{k=0}^{n}\binom{n-m-1}{k}(-1)^{n-k}a_{n-k}=\pm a_n\iff\sum_{k=0}^{n}\binom{n}{k}(-1)^k A_k=\pm A_n. \tag{2.9}$$

This together with the fact that

$$\begin{aligned}&\sum_{k=0}^{n+m+1}\binom{n+m+1-m-1}{k}(-1)^{n+m+1-k}a_{n+m+1-k}\\&=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k+m+1}a_{n-k+m+1}\\&=\sum_{r=0}^{n}\binom{n}{r}(-1)^{r+m+1}a_{r+m+1}\qquad(n=0,1,2,\dots)\end{aligned}$$

yields the result.

Lemma 2.3. *Let $\{a_n\}$ be a given sequence, $a(x)=\sum_{n=0}^{\infty}a_nx^n$ and $m\in\mathbb{R}$. Then*

$$\begin{aligned}&(1-x)^m a\Big(\frac{x}{x-1}\Big)=\pm a(x)\\&\iff\sum_{k=0}^{n}\binom{n-m-1}{k}(-1)^{n-k}a_{n-k}=\pm a_n\ (n=0,1,2,\dots).\end{aligned}$$

Proof. Clearly, for $|x|<1$,

$$(1-x)^m a\Big(\frac{x}{x-1}\Big)=\sum_{r=0}^{\infty}(-1)^r a_r x^r(1-x)^{m-r}$$

$$= \sum_{r=0}^{\infty}(-1)^r a_r x^r \sum_{k=0}^{\infty}\binom{m-r}{k}(-x)^k$$
$$= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}(-1)^{n-k}a_{n-k}\binom{m-(n-k)}{k}(-1)^k\Big)x^n$$
$$= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n-m-1}{k}(-1)^{n-k}a_{n-k}\Big)x^n.$$

Thus the result follows.

Theorem 2.3. *Let* $m \in \mathbb{N}$, $P_m(x) = \sum_{k=0}^m a_k x^{m-k}$ *and* $P_m^*(x) = \sum_{k=0}^m a_k x^k$. *Then the following statements are equivalent:*

(a) $(1-x)^m P_m^*\big(\frac{x}{x-1}\big) = \pm P_m^*(x)$.

(b) $P_m(1-x) = \pm(-1)^m P_m(x)$.

(c) *For* $n = 0,1,\dots,m$ *we have* $\sum_{k=0}^n \binom{n-m-1}{k}(-1)^{n-k}a_{n-k} = \pm a_n$.

(d) *Set* $a_n = 0$ *for* $n > m$. *Then* $\sum_{k=0}^n \binom{n-m-1}{k}(-1)^{n-k}a_{n-k} = \pm a_n$ $(n = 0,1,2,\dots)$.

(e) *For* $n = 0,1,\dots,m$ *we have*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{a_k}{\binom{m}{k}} = \pm(-1)^n \frac{a_n}{\binom{m}{n}}.$$

Proof. Since $P_m^*(x) = x^m P_m(\frac{1}{x})$ we see that

$$(1-x)^m P_m^*\Big(\frac{x}{x-1}\Big) = \pm P_m^*(x) \iff (-x)^m P_m\Big(1-\frac{1}{x}\Big) = \pm x^m P_m\Big(\frac{1}{x}\Big)$$
$$\iff (-1)^m P_m\Big(1-\frac{1}{x}\Big) = \pm P_m\Big(\frac{1}{x}\Big)$$
$$\iff P_m(1-x) = \pm(-1)^m P_m(x).$$

So (a) and (b) are equivalent. By Lemma 2.3, (a) is equivalent to (d). Assume $a_{n+m+1} = 0$ for $n \ge 0$. Then

$$a_{n+m+1} = 0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k+m+1}a_{m+1+n-k}$$
$$= \sum_{k=0}^{m+n+1}\binom{m+n+1-m-1}{k}(-1)^{m+n+1-k}a_{m+n+1-k}.$$

So (c) is equivalent to (d). To complete the proof, we note that (d) is equivalent to (e) by Lemma 2.2.

Remark 2.1. Let $\{B_n(x)\}$ and $\{E_n(x)\}$ be the Bernoulli polynomials and Euler polynomials given by

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k} \quad \text{and} \quad E_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(2x-1)^{n-k}E_k.$$

It is well known that ([4]) $B_n(1-x) = (-1)^n B_n(x)$ and $E_n(1-x) = (-1)^n E_n(x)$.

Theorem 2.4. *Let* $\{A_n\}\in S^+$ *with* $A_0=\ldots=A_{l-1}=0$ *and* $A_l\neq 0$ $(l\geq 1)$. *Then*

$$\Big\{\frac{A_{n+l}}{(n+1)(n+2)\cdots(n+l)}\Big\}\in S^+.$$

Proof. Assume $a_n=A_{n+l}$. Let $a(x)=\sum_{n=0}^\infty a_nx^n$ and $A(x)=\sum_{n=0}^\infty A_nx^n$. Then clearly $A(x)=x^la(x)$. Since $A_l=\sum_{k=0}^l\binom lk(-1)^kA_k=(-1)^lA_l$ we see that $2\mid l$. Thus, applying Lemma 2.3 and (2.9) we see that

$$\begin{aligned}\{A_n\}\in S^+ &\Leftrightarrow A(\frac{x}{x-1})=(1-x)A(x)\Leftrightarrow a\Big(\frac{x}{x-1}\Big)=(1-x)^{l+1}a(x)\\ &\Leftrightarrow \sum_{k=0}^n\binom{n+l}{k}(-1)^{n-k}a_{n-k}=a_n\ (n=0,1,2,\ldots)\Leftrightarrow\Big\{\frac{(-1)^na_n}{\binom{-l-1}{n}}\Big\}\in S^+.\end{aligned}$$

Note that

$$(-1)^n\frac{a_n}{\binom{-l-1}{n}}=\frac{a_n}{\binom{n+l}{l}}=\frac{A_{n+l}}{(n+1)(n+2)\cdots(n+l)}\cdot l!.$$

We then obtain the result.

Corollary 2.1. *Suppose that* $\{a_n\}\in S^+$ *with* $a_0\neq 0$ *and* $A_n=\frac{1}{(n+1)(n+2)}\sum_{k=0}^na_k$ $(n\geq 0)$. *Then* $\{A_n\}\in S^+$.

Proof. Let $b_0=b_1=0$ and $b_n=\sum_{k=0}^{n-2}a_k$ $(n\geq 2)$. By [10, Corollary 3.2], $\{b_n\}\in S^+$. Now applying Theorem 2.4 we find that $\{\frac{b_{n+2}}{(n+1)(n+2)}\}\in S^+$. That is, $\{A_n\}\in S^+$.

Theorem 2.5. *Let* F *be a given function. If* $\{A_n\}$ *is an even sequence, then*

$$\sum_{k=0}^n\binom nk(-1)^kA_k\Big(\sum_{s=0}^k\binom ks(-1)^s(F(s)-F(n-s))\Big)=0\quad(n=0,1,2,\ldots).$$

If $\{A_n\}$ *is an odd sequence, then*

$$\sum_{k=0}^n\binom nk(-1)^kA_k\Big(\sum_{s=0}^k\binom ks(-1)^s(F(s)+F(n-s))\Big)=0\quad(n=0,1,2,\ldots).$$

Proof. Suppose that $\sum_{k=0}^n\binom nk(-1)^kA_k=\pm A_n$ for $n=0,1,2,\ldots$. From [9, Lemma 2.1] we have

$$\sum_{k=0}^n\binom nk(-1)^kf(k)A_k=\pm\sum_{k=0}^n\binom nk\Big(\sum_{r=0}^k\binom kr(-1)^rF(n-k+r)\Big)A_k,$$

where $f(k)=\sum_{s=0}^k\binom ks(-1)^sF(s)$. Thus

$$\sum_{k=0}^n\binom nk(-1)^kA_k\Big(f(k)\mp\sum_{s=0}^k\binom ks(-1)^sF(n-s)\Big)=0.$$

This yields the result.

Corollary 2.2. *If* $\{A_n\}\in S^+$, *then*

$$\sum_{k=0}^n\binom nk(-1)^kA_k(1+x)^k(1-(-1)^nx^{n-k})=0\quad for\quad n=0,1,2,\ldots.$$

If $\{A_n\} \in S^-$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k A_k (1+x)^k (1+(-1)^n x^{n-k}) = 0 \quad \textit{for} \quad n = 0, 1, 2, \ldots.$$

Proof. Taking $F(s) = (-x)^s$ in Theorem 2.5 and then applying the binomial theorem we obtain the result.

Remark 2.2. From [9, (2.5)] we know that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(m+k) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k F(n+k),$$

where $F(r) = \sum_{s=0}^{r} \binom{r}{s} (-1)^s f(s)$. Hence for any nonnegative integers m and n we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k A_{k+m} = \sum_{k=0}^{m} \binom{m}{k} (-1)^k A_{k+n} \quad \text{for} \quad \{A_k\} \in S^+, \tag{2.10}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k A_{k+m} = -\sum_{k=0}^{m} \binom{m}{k} (-1)^k A_{k+n} \quad \text{for} \quad \{A_k\} \in S^-. \tag{2.11}$$

3. A transformation formula for $\sum_{k=0}^{n} \binom{n}{k} A_n$

Lemma 3.1 ([10, Theorems 4.1 and 4.2]). *Let f be a given function and $n \in \mathbb{N}$.*

(i) *If $\{A_n\}$ is an even sequence, then*

$$\sum_{k=0}^{n} \binom{n}{k} \Big(f(k) - (-1)^{n-k} \sum_{s=0}^{k} \binom{k}{s} f(s) \Big) A_{n-k} = 0.$$

(ii) *If $\{A_n\}$ is an odd sequence, then*

$$\sum_{k=0}^{n} \binom{n}{k} \Big(f(k) + (-1)^{n-k} \sum_{s=0}^{k} \binom{k}{s} f(s) \Big) A_{n-k} = 0.$$

We remark that a simple proof of Lemma 3.1 was given by Wang[13].

Theorem 3.1. *Let $n \in \mathbb{N}$. If $\{A_m\}$ is an even sequence and n is odd, or if $\{A_m\}$ is an odd sequence and n is even, then*

$$\sum_{\substack{k=0 \\ 3|k}}^{n} \binom{n}{k} A_{n-k} = \sum_{\substack{k=0 \\ 3|n-k}}^{n} \binom{n}{k} A_k = \frac{1}{3} \sum_{k=0}^{n} \binom{n}{k} A_k.$$

Proof. Set $\omega = (-1+\sqrt{-3})/2$. If $\{A_m\}$ is an even sequence and n is odd, or if $\{A_m\}$ is an odd sequence and n is even, putting $f(k) = \omega^k$ in Lemma 3.1 we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (\omega^k + (-1)^k (1+\omega)^k) A_{n-k} = 0.$$

As $1+\omega=-\omega^2$, we have $\sum_{k=0}^{n}\binom{n}{k}(\omega^k+\omega^{2k})A_{n-k}=0$. Therefore,

$$\begin{aligned}
&3\sum_{\substack{k=0\\3\mid k}}^{n}\binom{n}{k}A_{n-k}-\sum_{k=0}^{n}\binom{n}{k}A_k\\
&=\sum_{k=0}^{n}\binom{n}{k}(1+\omega^k+\omega^{2k})A_{n-k}-\sum_{k=0}^{n}\binom{n}{k}A_{n-k}\\
&=\sum_{k=0}^{n}\binom{n}{k}(\omega^k+\omega^{2k})A_{n-k}=0.
\end{aligned}$$

This proves the theorem.

Corollary 3.1 (Ramanujan [7]). *For $n=3,5,7,\dots$ we have*

$$\sum_{\substack{k=0\\6\mid k-3}}^{n}\binom{n}{k}B_{n-k}=\begin{cases}-\dfrac{n}{6} & \text{if } n\equiv 1 \pmod 6,\\ \dfrac{n}{3} & \text{if } n\equiv 3,5 \pmod 6.\end{cases}$$

Proof. As $\{(-1)^nB_n\}\in S^+$, $B_1=-\frac{1}{2}$ and $B_{2m+1}=0$ for $m\ge 1$, taking $A_n=(-1)^nB_n$ in Theorem 3.1 we obtain

$$\begin{aligned}
\sum_{\substack{k=0\\3\mid k}}^{n}\binom{n}{k}(-1)^{n-k}B_{n-k}&=\frac{1}{3}\sum_{k=0}^{n}\binom{n}{k}(-1)^kB_k=\frac{1}{3}\Big(\sum_{k=0}^{n}\binom{n}{k}B_k+n\Big)\\
&=\frac{1}{3}(n+B_n)=\frac{n}{3}.
\end{aligned}$$

To see the result, we note that

$$\sum_{\substack{k=0\\3\mid k}}^{n}\binom{n}{k}(-1)^{n-k}B_{n-k}-\sum_{\substack{k=0\\6\mid k-3}}^{n}\binom{n}{k}B_{n-k}=\begin{cases}-nB_1=\dfrac{n}{2} & \text{if } n\equiv 1 \pmod 6,\\ 0 & \text{if } n\equiv 3,5 \pmod 6.\end{cases}$$

Corollary 3.2 (Ramanujan [7]). *For $n=0,2,4,\dots$ we have*

$$\frac{1}{3}(2^n-1)B_n+\sum_{k=0}^{[n/6]}\binom{n}{6k}(2^{n-6k}-1)B_{n-6k}=\begin{cases}-\dfrac{n}{6} & \text{if } n\equiv 4 \pmod 6,\\ \dfrac{n}{3} & \text{if } n\equiv 0,2 \pmod 6.\end{cases}$$

Proof. Since $\{(-1)^n(2^n-1)B_n\}$ is an odd sequence, $B_1=-\frac{1}{2}$ and $B_{2m+1}=0$ for $m\ge 1$, applying Theorem 3.1 we see that for even n,

$$\begin{aligned}
&\sum_{\substack{k=0\\3\mid k}}^{n}\binom{n}{k}(-1)^{n-k}(2^{n-k}-1)B_{n-k}\\
&=\frac{1}{3}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(2^k-1)B_k=\frac{1}{3}\Big(\sum_{k=0}^{n}\binom{n}{k}(2^k-1)B_k+n\Big)\\
&=\frac{1}{3}(-(-1)^n(2^n-1)B_n+n)=\frac{n}{3}-\frac{1}{3}(2^n-1)B_n.
\end{aligned}$$

On the other hand, for even n,

$$\sum_{\substack{k=0\\3|k}}^{n}\binom{n}{k}(-1)^{n-k}(2^{n-k}-1)B_{n-k}-\sum_{\substack{k=0\\6|k}}^{n}\binom{n}{k}(2^{n-k}-1)B_{n-k}$$
$$=\begin{cases}-nB_1=\dfrac{n}{2} & \text{if } 6\mid n-4,\\ 0 & \text{if } 6\nmid n-4.\end{cases}$$

Now combining all the above yields the result.

Corollary 3.3 (Lehmer [3]). *For $n=0,2,4,\ldots$ we have*

$$E_n+3\sum_{k=1}^{[n/6]}\binom{n}{6k}2^{6k-2}E_{n-6k}=\frac{1+(-3)^{n/2}}{2}.$$

Proof. Since $\{(E_n-1)/2^n\}$ is an odd sequence and $E_{2k+1}=0$, from Theorem 3.1 and the binomial theorem we see that for even n,

$$\sum_{\substack{k=0\\6|k}}^{n}\binom{n}{k}\frac{E_{n-k}}{2^{n-k}}-\sum_{\substack{k=0\\3|k}}^{n}\binom{n}{k}\frac{1}{2^{n-k}}=\sum_{\substack{k=0\\3|k}}^{n}\binom{n}{k}\frac{E_{n-k}-1}{2^{n-k}}=\frac{1}{3}\sum_{k=0}^{n}\binom{n}{k}\frac{E_k-1}{2^k}$$
$$=\frac{1}{3}\Big\{\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{E_k-1}{2^k}+\Big(1-\frac{1}{2}\Big)^n-\Big(1+\frac{1}{2}\Big)^n\Big\}$$
$$=\frac{1}{3}\Big\{-\frac{E_n-1}{2^n}+\frac{1-3^n}{2^n}\Big\}=\frac{2-3^n-E_n}{3\cdot 2^n}.$$

For even n we also have

$$\sum_{\substack{k=0\\3|k}}^{n}\binom{n}{k}2^k=\sum_{k=0}^{n}\binom{n}{k}2^k\cdot\frac{1}{3}(1+\omega^k+\omega^{2k})$$
$$=\frac{1}{3}\big((1+2)^n+(1+2\omega)^n+(1+2\omega^2)^n\big)$$
$$=\frac{1}{3}\big(3^n+(\sqrt{-3})^n+(-\sqrt{-3})^n\big)=\frac{1}{3}(3^n+2\cdot(-3)^{\frac{n}{2}}).$$

Hence

$$\frac{1}{3}E_n+\sum_{\substack{k=0\\6|k}}^{n}\binom{n}{k}2^kE_{n-k}=\frac{2-3^n}{3}+\frac{3^n+2\cdot(-3)^{\frac{n}{2}}}{3}=\frac{2}{3}\big(1+(-3)^{\frac{n}{2}}\big).$$

This yields the result.

Remark 3.1 Compared with known proofs of Corollaries 3.1-3.3 (see [1,3,7]), our proofs are simple and natural.

4. Congruences involving even and odd sequences

Let p be an odd prime. For $k\in\{1,2,\ldots,p-1\}$ we see that

$$\binom{p}{k}=\frac{p(p-1)\cdots(p-(k-1))}{k!}\equiv(-1)^{k-1}\frac{p}{k}\pmod{p^2}. \tag{4.1}$$

If $\{A_n\}$ is an even sequence and $A_0, A_1, \ldots, A_{p-2}, pA_{p-1}, A_p \in \mathbb{Z}_p$, applying (4.1) we see that

$$\begin{aligned} A_p &= \sum_{k=0}^{p} \binom{p}{k}(-1)^k A_k = A_0 + pA_{p-1} - A_p + \sum_{k=1}^{p-2} \binom{p}{k}(-1)^k A_k \\ &\equiv A_0 + pA_{p-1} - A_p - p\sum_{k=1}^{p-2} \frac{A_k}{k} \pmod{p^2}. \end{aligned}$$

Hence

$$(4.2) \qquad 2A_p - pA_{p-1} \equiv A_0 - p\sum_{k=1}^{p-2} \frac{A_k}{k} \pmod{p^2} \quad \text{for} \quad \{A_n\} \in S^+.$$

If $\{A_n\}$ is an odd sequence and $A_0, A_1, \ldots, A_p \in \mathbb{Z}_p$, using (4.1) we see that

$$-A_p = \sum_{k=0}^{p} \binom{p}{k}(-1)^k A_k \equiv A_0 - A_p - p\sum_{k=1}^{p-1} \frac{A_k}{k} \pmod{p^2}.$$

Since $A_0 = -A_0$ we have $A_0 = 0$ and so

$$(4.3) \qquad \sum_{k=1}^{p-1} \frac{A_k}{k} \equiv 0 \pmod{p} \quad \text{for} \quad \{A_n\} \in S^-.$$

We note that (4.3) was first obtained by Mattarei and Tauraso[5] via a complicated method.

For an odd prime p and $a \in \mathbb{Z}_p$ let $\langle a\rangle_p \in \{0, 1, \ldots p-1\}$ be given by $\langle a\rangle_p \equiv a \pmod{p}$. Let p be an odd prime, $a \in \mathbb{Z}_p$ and $A_0, A_1, \ldots, A_{p-1} \in \mathbb{Z}_p$. By [11, Theorem 2.4], if $\langle a\rangle_p$ is odd and $\{A_n\}$ is an even sequence, or if $\langle a\rangle_p$ is even and $\{A_n\}$ is an odd sequence, then

$$(4.4) \qquad \sum_{k=0}^{p-1} \binom{a}{k}\binom{-1-a}{k} A_k \equiv 0 \pmod{p^2}.$$

In the case $a = -\frac{1}{2}$, (4.4) was given by the author in an earlier unpublished preprint. Inspired by the author, Z.W. Sun deduced (4.4) in the cases $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$. See [12, Theorem 1.4].

Now we establish new congruences for sums involving even or odd sequences.

Theorem 4.1. *Let p be an odd prime. If $\{A_n\} \in S^+$ and $A_1, \ldots, A_{p-2}, pA_{p-1}, A_p \in \mathbb{Z}_p$, then*

$$A_p - \frac{pA_{p-1}}{2} \equiv (p+1)A_1 - p\sum_{k=1}^{p-3} \frac{A_{k+1}}{k} \pmod{p^2}.$$

If $\{A_n\} \in S^-$ and $A_1, A_2, \ldots, A_{p+1} \in \mathbb{Z}_p$, then

$$2A_{p+1} - A_p \equiv A_1 - p\sum_{k=1}^{p-1} \frac{A_{k+1}}{k} \pmod{p^2}.$$

Proof. If $\{A_n\} \in S^+$, from [10, Corollary 3.1(a)] we have $\{2A_{n+1} - A_n\} \in S^-$. Thus,

$$A_p - 2A_{p+1} = \sum_{k=0}^{p} \binom{p}{k}(-1)^k(2A_{k+1} - A_k) = 2\sum_{k=0}^{p} \binom{p}{k}(-1)^k A_{k+1} - A_p$$
$$= 2\Big(A_1 - A_{p+1} + \sum_{k=1}^{p-1} \binom{p}{k}(-1)^k A_{k+1}\Big) - A_p.$$

Hence applying (4.1) we deduce that

$$A_p = A_1 + pA_p - \frac{p(p-1)}{2}A_{p-1} + \sum_{k=1}^{p-3} \binom{p}{k}(-1)^k A_{k+1}$$
$$\equiv A_1 + pA_p - \frac{p(p-1)}{2}A_{p-1} - p\sum_{k=1}^{p-3} \frac{A_{k+1}}{k} \pmod{p^2}.$$

This yields the first part. If $\{A_n\} \in S^-$, from [10, Corollary 3.1(a)] we have $\{2A_{n+1} - A_n\} \in S^+$. Thus,

$$2A_{p+1} - A_p = \sum_{k=0}^{p} \binom{p}{k}(-1)^k(2A_{k+1} - A_k) = 2\sum_{k=0}^{p} \binom{p}{k}(-1)^k A_{k+1} + A_p$$
$$= 2\Big(A_1 - A_{p+1} + \sum_{k=1}^{p-1} \binom{p}{k}(-1)^k A_{k+1}\Big) + A_p.$$

Hence applying (4.1) we obtain

$$A_{p+1} - A_p = A_1 - A_{p+1} + \sum_{k=1}^{p-1} \binom{p}{k}(-1)^k A_{k+1} \equiv A_1 - A_{p+1} - p\sum_{k=1}^{p-1} \frac{A_{k+1}}{k} \pmod{p^2}.$$

This yields the remaining result. The proof is now complete.

Corollary 4.1. *Let p be an odd prime. Then*

$$pB_{p-1} \equiv -p - 1 + 2p\sum_{k=1}^{(p-3)/2} \frac{B_{2k}}{2k-1} \pmod{p^2}.$$

Proof. Since $\{(-1)^n B_n\} \in S^+$, $B_1 = -\frac{1}{2}$ and $B_{2m+1} = 0$ for $m > 1$, taking $A_n = (-1)^n B_n$ in Theorem 4.1 yields the result.

Corollary 4.2. *Let p be an odd prime and $b, c \in \mathbb{Z}_p$ with $b \not\equiv 0 \pmod{p}$. Then*

$$V_p(b,c) \equiv b^p\Big(1 - p\sum_{k=1}^{p-1} \frac{U_{k+1}(b,c)}{kb^k}\Big) \pmod{p^2}.$$

Proof. Since $\{\frac{U_n(b,c)}{b^n}\} \in S^-$ and $V_p(b,c) = 2U_{p+1}(b,c) - bU_p(b,c)$, from Theorem 4.1 we deduce the result.

Theorem 4.2. *Let p be an odd prime . If $\{A_n\} \in S^+$ and $A_0, A_1, \ldots, A_{p-2} \in \mathbb{Z}_p$, then*

$$\sum_{k=0}^{p-2} A_k \equiv p\sum_{k=0}^{p-2} \frac{A_k}{k+1} \pmod{p^2}.$$

If $\{A_n\}\in S^-$ *and* $A_0,A_1,\ldots,A_{p-1}\in\mathbb{Z}_p$, *then*

$$\sum_{k=0}^{p-2}A_k\equiv -2A_{p-1}-p\sum_{k=0}^{p-2}\frac{A_k}{k+1}\pmod{p^2}.$$

Proof. Since

$$\sum_{k=0}^{p-1}\binom{p-1-p}{k}(-1)^{p-1-k}A_{p-1-k}=\sum_{k=0}^{p-1}A_{p-1-k}=\sum_{k=0}^{p-1}A_k$$

and

$$\sum_{k=0}^{p-1}\binom{p}{k}(-1)^kA_{p-1-k}=A_{p-1}+\sum_{k=0}^{p-2}\binom{p}{k+1}(-1)^kA_k,$$

putting $m=0$ and $n=p-1$ in Lemma 2.1 and then applying (4.1) we see that if $\{A_n\}\in S^+$, then

$$\sum_{k=0}^{p-2}A_k=\sum_{k=0}^{p-2}\binom{p}{k+1}(-1)^kA_k\equiv p\sum_{k=0}^{p-2}\frac{A_k}{k+1}\pmod{p^2};$$

if $\{A_n\}\in S^-$, then

$$\sum_{k=0}^{p-1}A_k=-A_{p-1}-\sum_{k=0}^{p-2}\binom{p}{k+1}(-1)^kA_k\equiv -A_{p-1}-p\sum_{k=0}^{p-2}\frac{A_k}{k+1}\pmod{p^2}.$$

This yields the result.

Corollary 4.3. *Let* $p>3$ *be a prime and* $b,c\in\mathbb{Z}_p$ *with* $bc\not\equiv 0\pmod p$. *Then*

$$V_p(b,c)\equiv b^p-\frac{pc}{b}\sum_{k=0}^{p-2}\frac{V_k(b,c)}{(k+1)b^k}\pmod{p^2}.$$

Proof. It is well known that $V_k(b,c)=\big(\frac{b+\sqrt{b^2-4c}}{2}\big)^k+\big(\frac{b-\sqrt{b^2-4c}}{2}\big)^k$. Thus,

$$\begin{aligned}\sum_{k=0}^{p-2}\frac{V_k(b,c)}{b^k}&=\sum_{k=0}^{p-2}\Big(\frac{b+\sqrt{b^2-4c}}{2b}\Big)^k+\sum_{k=0}^{p-2}\Big(\frac{b-\sqrt{b^2-4c}}{2b}\Big)^k\\&=\frac{1-(\frac{b+\sqrt{b^2-4c}}{2b})^{p-1}}{1-\frac{b+\sqrt{b^2-4c}}{2b}}+\frac{1-(\frac{b-\sqrt{b^2-4c}}{2b})^{p-1}}{1-\frac{b-\sqrt{b^2-4c}}{2b}}\\&=\frac{4b^2}{4c}\Big(1-\frac{V_p(b,c)}{b^p}\Big)=\frac{b^p-V_p(b,c)}{b^{p-2}c}.\end{aligned}$$

Since $\{\frac{V_n(b,c)}{b^n}\}\in S^+$, from the above and Theorem 4.2 we deduce the result.

Theorem 4.3. *Let* p *be an odd prime, and let* $\{A_n\}$ *be an odd sequence. Suppose* $A_1,A_2,\ldots,A_{p-1}\in\mathbb{Z}_p$. *Then*

$$\sum_{k=1}^{p-1}\frac{A_k}{k}\equiv p\sum_{k=1}^{p-1}\frac{A_k}{k^2}\pmod{p^2}.$$

Proof. Taking $n=p-1$ in Theorem 2.2 we get $\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p-1+k}{k}(-1)^kA_k=0$. For $k=1,2,\ldots,p-1$ we see that

$$\begin{aligned}\binom{p-1}{k}\binom{p-1+k}{k}&=\frac{(p-1)(p-2)\cdots(p-k)}{k!}\cdot\frac{p(p+1)\cdots(p+k-1)}{k!}\\&=\frac{p}{p+k}\cdot\frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-k^2)}{k!^2}\equiv(-1)^k\frac{p}{p+k}\pmod{p^3}.\end{aligned}$$

Since $A_0=-A_0$ we have $A_0=0$. Now, from all the above we deduce that

$$\sum_{k=1}^{p-1}\frac{A_k}{p+k}\equiv 0\pmod{p^2}\quad for\quad \{A_n\}\in S^-. \tag{4.5}$$

For $k=1,2,\ldots,p-1$ we have $\frac{1}{k+p}=\frac{k-p}{k^2-p^2}\equiv\frac{k-p}{k^2}=\frac1k-\frac{p}{k^2}\pmod{p^2}$. Thus the result follows.

Let $F_n=U_n(1,-1)$ and $L_n=V_n(1,-1)$ be the Fibonacci and Lucas sequences, respectively. From Theorem 4.3 we have the following corollary.

Corollary 4.4. *Let $p>5$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{F_k}{k}\equiv-\Big(\frac p5\Big)\frac{5p}{4}\Big(\frac{F_{p-(\frac p5)}}{p}\Big)^2\pmod{p^2}.$$

Proof. Recently Pan and Sun ([6, Remark 3.3]) proved that

$$\sum_{k=1}^{p-1}\frac{F_k}{k^2}\equiv-\frac15\Big(\frac p5\Big)\Big(\frac{L_p-1}{p}\Big)^2\pmod{p}.$$

It is known that ([8]) $F_{p-(\frac p5)}\equiv 0\pmod p$ and $L_{p-(\frac p5)}\equiv 2(\frac p5)\pmod{p^2}$. Also, $5F_n=2L_{n+1}-L_n=L_n+2L_{n-1}$. Thus

$$5F_{p-(\frac p5)}=2L_p-\Big(\frac p5\Big)L_{p-(\frac p5)}\equiv 2(L_p-1)\pmod{p^2}$$

and so

$$\sum_{k=1}^{p-1}\frac{F_k}{k^2}\equiv-\frac15\Big(\frac p5\Big)\Big(\frac{5F_{p-(\frac p5)}}{2p}\Big)^2\pmod{p^2}.$$

Since $\{F_k\}$ is an odd sequence, applying Theorem 4.3 we deduce the result.

Theorem 4.4. *Let p be a prime greater than 3, and let $\{A_n\}$ be an even sequence. Suppose that $A_0,A_1,\ldots,A_{p-2},A_p,pA_{p-1}\in\mathbb{Z}_p$. Then*

$$\sum_{k=1}^{p-2}\frac{A_k}{k}\equiv-p\sum_{k=1}^{p-2}\frac{A_k}{k^2}+\frac{A_0+pA_{p-1}-2A_p}{p}\pmod{p^2}.$$

Proof. Taking $n=p$ in Theorem 2.2 we get $\sum_{k=0}^{p}\binom pk\binom{p+k}{k}(-1)^kA_k=0$. For $k=1,2,\ldots,p-1$ we see that

$$\binom pk\binom{p+k}{k}=\frac{p(p-1)\cdots(p-k+1)}{k!}\cdot\frac{(p+1)\cdots(p+k)}{k!}$$

$$= \frac{p}{p-k} \cdot \frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-k^2)}{k!^2} \equiv (-1)^k \frac{p}{p-k} \pmod{p^3}.$$

Thus,

$$A_0 - \binom{2p}{p} A_p + \binom{p}{p-1}\binom{2p-1}{p-1} A_{p-1} + \sum_{k=1}^{p-2} \frac{p}{p-k} A_k$$
$$\equiv \sum_{k=0}^{p} \binom{p}{k}\binom{p+k}{k} (-1)^k A_k = 0 \pmod{p^3}.$$

Hence

$$\sum_{k=1}^{p-2} \frac{A_k}{p-k} \equiv \frac{2\binom{2p-1}{p-1} A_p - p\binom{2p-1}{p-1} A_{p-1} - A_0}{p} \pmod{p^2}.$$

The famous Wolstenholme's congruence ([15]) states that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$. Thus

(4.6) $$\sum_{k=1}^{p-2} \frac{A_k}{p-k} \equiv \frac{2A_p - A_0 - pA_{p-1}}{p} \pmod{p^2} \quad for \quad \{A_n\} \in S^+.$$

For $k = 1, 2, \ldots, p-2$ we have $\frac{1}{k-p} = \frac{k+p}{k^2-p^2} \equiv \frac{k+p}{k^2} = \frac{1}{k} + \frac{p}{k^2} \pmod{p^2}$. Hence the result follows.

Corollary 4.5. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{L_k}{k} \equiv \frac{2(1-L_p)}{p} \pmod{p^2}.$$

Proof. In [6] Pan and Sun proved that $\sum_{k=1}^{p-1} \frac{L_k}{k^2} \equiv 0 \pmod{p}$, which was conjectured by R. Tauraso. Since $\{L_n\} \in S^+$, taking $A_k = L_k$ in Theorem 4.4 we see that

$$\sum_{k=1}^{p-2} \frac{L_k}{k} \equiv -p\Big(\sum_{k=1}^{p-1} \frac{L_k}{k^2} - \frac{L_{p-1}}{(p-1)^2}\Big) + \frac{2 + pL_{p-1} - 2L_p}{p}$$
$$\equiv (p+1)L_{p-1} + \frac{2(1-L_p)}{p} \equiv -\frac{L_{p-1}}{p-1} + \frac{2(1-L_p)}{p} \pmod{p^2}.$$

This yields the result.

Corollary 4.6. *Let p be a prime greater than 3. Then*

$$\sum_{k=1}^{(p-3)/2} \frac{B_{2k}}{p-2k} \equiv \frac{p+1}{2} - \frac{pB_{p-1}+1}{p} \pmod{p^2}.$$

Proof. It is well known that $B_0, B_1, \ldots, B_{p-2}, B_p, pB_{p-1} \in \mathbb{Z}_p$. Taking $A_k = (-1)^k B_k$ in (4.6) and applying the fact $B_{2k+1} = 0$ for $k \geq 1$ we deduce the result.

Corollary 4.7. *Let $p > 3$ be a prime, $b, c \in \mathbb{Z}_p$ and $b \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=1}^{p-1} \frac{V_k(b,c)}{(p-k)b^k} \equiv \frac{2(V_p(b,c) - b^p)}{pb^p} \pmod{p^2}.$$

Proof. Taking $A_k = V_k(b,c)/b^k$ in (4.6) yields the result.

Theorem 4.5. *Let p be an odd prime and $A_0, A_1, \dots, A_{\frac{p-1}{2}} \in \mathbb{Z}_p$. If $\{A_n\} \in S^+$ and $p \equiv 3 \pmod 4$, or if $\{A_n\} \in S^-$ and $p \equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{A_k}{2^k} \equiv 0 \pmod p.$$

Proof. Since $\{\frac{1}{2^n}\} \in S^+$, by Lemma 3.1(i) we have

$$\sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} (-1)^k A_k \frac{2}{2^{\frac{p-1}{2}-k}}$$
$$= \sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} \Big((-1)^k A_k - (-1)^{\frac{p-1}{2}-k} \sum_{s=0}^{k} \binom{k}{s} (-1)^s A_s \Big) \frac{1}{2^{\frac{p-1}{2}-k}} = 0.$$

Note that $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac12}{k} = \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$ by [2, p.90]. From the above we deduce the result.

Theorem 4.6. *Let p be an odd prime and $A_0, A_1, \dots, A_p \in \mathbb{Z}_p$. If $\{A_n\}$ is an odd sequence, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k} A_{\frac{p-1}{2}-k} \equiv -(-1)^{\frac{p-1}{2}} A_{\frac{p-1}{2}} + (-1)^{\frac{p-1}{2}} \frac{p}{2} \sum_{k=1}^{(p-1)/2} \frac{A_{\frac{p-1}{2}-k}}{k} \pmod {p^2}$$

and

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{A_{p-1-k}}{4^k} \equiv 0 \pmod p.$$

If $\{A_n\}$ is an even sequence, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k} A_{\frac{p-1}{2}-k} \equiv (-1)^{\frac{p-1}{2}} A_{\frac{p-1}{2}} - (-1)^{\frac{p-1}{2}} \frac{p}{2} \sum_{k=1}^{(p-1)/2} \frac{A_{\frac{p-1}{2}-k}}{k} \pmod {p^2}$$

and

$$\frac{A_p - A_0/2}{p} \equiv -A_0 \frac{2^{p-1}-1}{p} + \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k \cdot k} A_{p-k} \pmod p.$$

Proof. Putting $m = 0$, $p = -\frac12$ in Lemma 2.1 and noting that $\binom{-\frac12}{k} = \binom{2k}{k}(-4)^{-k}$ we see that if $\sum_{k=0}^{n} \binom{n}{k}(-1)^k A_k = \pm A_n$ for $n \ge 0$, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k} A_{\frac{p-1}{2}-k} = \sum_{k=0}^{(p-1)/2} \binom{-\frac12}{k} (-1)^k A_{\frac{p-1}{2}-k} = \pm \sum_{k=0}^{(p-1)/2} \binom{\frac{p}{2}}{k} (-1)^{\frac{p-1}{2}-k} A_{\frac{p-1}{2}-k}$$
$$\equiv \pm (-1)^{\frac{p-1}{2}} \Big(A_{\frac{p-1}{2}} - \sum_{k=1}^{(p-1)/2} \frac{p}{2k} A_{\frac{p-1}{2}-k} \Big) \pmod {p^2},$$

where in the last step we use the fact $\binom{ap}{k} = \frac{ap}{k}\binom{ap-1}{k-1} \equiv \frac{ap}{k}\binom{-1}{k-1} = (-1)^{k-1}\frac{ap}{k} \pmod{p^2}$ for $1 \le k \le p-1$.

Note that $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$. If $\{A_n\}$ is an odd sequence, taking $n = \frac{p-1}{2}$ in Theorem 2.1 and applying the above we deduce that $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{A_{p-1-k}}{4^k} \equiv 0 \pmod p$. Now assume that $\{A_n\}$ is an even sequence. Since $p \mid \binom{p}{k}$ for $k = 1, 2, \dots, p-1$, we see that $A_p = \sum_{k=0}^{p}\binom{p}{k}(-1)^k A_k \equiv A_0 - A_p \pmod p$ and so $A_p \equiv A_0/2 \pmod p$. By Theorem 2.1,

$$\frac{A_p}{p} + \sum_{k=1}^{(p-1)/2}\binom{\frac{p-1}{2}}{k}(-1)^k\frac{A_{p-k}}{p-k} = (-1)^{\frac{p-1}{2}}\frac{A_0/2}{p\binom{p-1}{(p-1)/2}}.$$

Since $\binom{(p-1)/2}{k} \equiv \binom{2k}{k}/(-4)^k \pmod p$, we deduce that

$$\frac{A_p\binom{p-1}{(p-1)/2} - (-1)^{(p-1)/2}A_0/2}{p\binom{p-1}{(p-1)/2}} \equiv \sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}A_{p-k}}{4^k \cdot k} \pmod p.$$

It is well known that (see [8, Corollary 1.2] or [9, Theorem 5.2]) $\sum_{k=1}^{(p-1)/2}\frac{1}{k} \equiv -\frac{2^p-2}{p} \pmod p$. Thus,

$$\begin{aligned}\binom{p-1}{\frac{p-1}{2}} &= \frac{(p-1)(p-2)\cdots(p-\frac{p-1}{2})}{\frac{p-1}{2}!} \equiv (-1)^{\frac{p-1}{2}}\Big(1 - p\sum_{k=1}^{(p-1)/2}\frac{1}{k}\Big)\\ &\equiv (-1)^{\frac{p-1}{2}}(2^p-1) \pmod{p^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{A_p\binom{p-1}{(p-1)/2} - (-1)^{(p-1)/2}A_0/2}{p\binom{p-1}{(p-1)/2}} &\equiv \frac{A_p(1+2^p-2) - A_0/2}{p(1+2^p-2)} \equiv \frac{A_p - A_0/2}{p} + \frac{2^p-2}{p}A_p\\ &\equiv \frac{A_p - A_0/2}{p} + A_0\frac{2^{p-1}-1}{p} \pmod p.\end{aligned}$$

Combining all the above proves the theorem.

Added Remark. Let p be an odd prime and $A_0, A_1, \dots, A_{\frac{p-1}{2}} \in \mathbb{Z}_p$. Observe that $\binom{-\frac{1}{2}}{k} = \binom{2k}{k}(-4)^{-k}$ and $\binom{p/2}{k} = \frac{p}{2k}\binom{p/2-1}{k-1} \equiv \frac{p}{2k}\binom{-1}{k-1} = -\frac{(-1)^k}{2k}p \pmod{p^2}$ for $k \in \mathbb{N}$. Putting $m = 0$, $p = -\frac{1}{2}$, $n = \frac{p-1}{2}$ in Lemma 2.1 and then applying the above we deduce that if $\sum_{k=0}^{n}\binom{n}{k}(-1)^k A_k = \pm A_n$ for $n \ge 0$, then

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{4^k}A_{\frac{p-1}{2}-k} &= \pm\sum_{k=0}^{(p-1)/2}\binom{\frac{p}{2}}{k}(-1)^{\frac{p-1}{2}-k}A_{\frac{p-1}{2}-k}\\ &\equiv \pm(-1)^{\frac{p-1}{2}}\Big(A_{\frac{p-1}{2}} - \frac{p}{2}\sum_{k=1}^{(p-1)/2}\frac{A_{\frac{p-1}{2}-k}}{k}\Big) \pmod{p^2}.\end{aligned}$$

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