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ABSTRACT. Let $p > 3$ be a prime, and let m be an integer with $p \nmid m$. In the paper we solve some conjectures of Z.W. Sun concerning $\sum_{k=0}^{p-1} \binom{2k}{k}^3 / m^k \pmod{p^2}$, $\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} / m^k \pmod{p}$ and $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} / m^k \pmod{p^2}$. In particular, we show that $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \equiv 0 \pmod{p^2}$ for $p \equiv 3, 5, 6 \pmod{7}$. Let $\{P_n(x)\}$ be the Legendre polynomials. In the paper we also show that $P_{[\frac{p}{4}]}(t) \equiv -\binom{6}{p} \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3}{2}(3t+5)x + 9t+7}{p} \right) \pmod{p}$, where t is a rational p -adic integer, $[x]$ is the greatest integer not exceeding x and $\left(\frac{a}{p}\right)$ is the Legendre symbol. As consequences we determine $P_{[\frac{p}{4}]}(t) \pmod{p}$ in the cases $t = -\frac{5}{3}, -\frac{7}{9}, -\frac{65}{63}$ and confirm many conjectures of Z.W. Sun.

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1. Introduction.

Let p be an odd prime. Following [A] we define

$$A(p, \lambda) = \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 \binom{\frac{p-1}{2} + k}{k} \lambda^{kp}.$$

It is easily seen that (see [S2, Lemmas 2.2, 2.4 and 2.5]) for $k \in \{0, 1, \dots, \frac{p-1}{2}\}$,

$$(1.1) \quad (-1)^k \binom{\frac{p-1}{2} + k}{k} \equiv \binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod{p}, \quad \binom{\frac{p-1}{2} + k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}.$$

Thus, by Fermat's little theorem, for any rational p -adic integer λ ,

$$(1.2) \quad A(p, \lambda) \equiv \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \left(\frac{\lambda}{64}\right)^k \pmod{p}.$$

For positive integers a, b and n , if $n = ax^2 + by^2$ for some integers x and y , we briefly say that $n = ax^2 + by^2$. In 1987, Beukers[B] conjectured a congruence for $A(p, 1) \pmod{p^2}$ equivalent to

$$(1.3) \quad \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 4y^2 \equiv 1 \pmod{4}. \end{cases}$$

This congruence was proved by several authors including Ishikawa[Is]($p \equiv 1 \pmod{4}$), van Hamme[H]($p \equiv 3 \pmod{4}$) and Ahlgren[A]. In 1998, by using the hypergeometric series ${}_3F_2(\lambda)_p$ over the finite field \mathbb{F}_p , Ono[O] obtained congruences for $A(p, \lambda) \pmod{p}$ in the cases $\lambda = -1, -8, -\frac{1}{8}, 4, \frac{1}{4}, 64, \frac{1}{64}$, see also [A]. Hence from (1.2) we deduce congruences for $\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{m^k} \binom{2k}{k}^3 \pmod{p}$ in the cases $m = 1, -8, 16, -64, 256, -512, 4096$. Recently the author's brother Zhi-Wei Sun[Su1,Su2,Su3] conjectured corresponding congruences modulo p^2 . In particular, he conjectured that

$$(1.4) \quad \sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 7y^2 \equiv 1, 2, 4 \pmod{7}. \end{cases}$$

We note that $p \mid \binom{2k}{k}$ for $\frac{p+1}{2} \leq k \leq p-1$. In the paper, by using Legendre polynomials and the work of Coster and van Hamme[CH] we completely solve Zhi-Wei Sun's such conjectures.

Let $\{P_n(x)\}$ be the Legendre polynomials given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad (n \geq 1).$$

It is well known that (see [MOS, pp. 228-232], [G, (3.132)-(3.133)])

$$(1.5) \quad P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (-1)^k \binom{2n-2k}{n} x^{n-2k} = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n,$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x . From (1.5) we see that

$$(1.6) \quad P_n(-x) = (-1)^n P_n(x), \quad P_{2m+1}(0) = 0 \quad \text{and} \quad P_{2m}(0) = \frac{(-1)^m}{2^{2m}} \binom{2m}{m}.$$

The first few Legendre polynomials are as follows:

$$\begin{aligned} P_0(x) &= 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned}$$

Let \mathbb{Z} be the set of integers. For a prime p , let \mathbb{Z}_p be the set of those rational numbers whose denominator is not divisible by p , let \mathbb{Q}_p denote the field of p -adic numbers, and let $\left(\frac{a}{p}\right)$ be the Legendre symbol. On the basis of the work of Ono[O] (see also [A, Theorem 2] and [LR]), in [S2, Theorem 2.11] the author showed that for any prime $p > 3$ and $t \in \mathbb{Z}_p$,

$$(1.7) \quad P_{\frac{p-1}{2}}(t) \equiv -\left(\frac{-6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 3(t^2 + 3)x + 2t(t^2 - 9)}{p}\right) \pmod{p}.$$

In this paper, by using elementary arguments we prove that for any prime $p > 3$ and $t \in \mathbb{Z}_p$,

$$(1.8) \quad P_{[\frac{p}{4}]}(t) \equiv -\left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3}{2}(3t+5)x + 9t+7}{p} \right) \pmod{p}.$$

From (1.8) we deduce (1.7) immediately. As consequences of (1.8), we determine $P_{[\frac{p}{4}]}(-\frac{5}{3})$, $P_{[\frac{p}{4}]}(-\frac{7}{9})$, $P_{[\frac{p}{4}]}(-\frac{65}{63}) \pmod{p}$ and use them to solve Z.W. Sun's conjectures ([Su1]) on

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{m^k} \pmod{p} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \pmod{p^2}.$$

For instance, for any prime $p > 7$,

$$(1.9) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{81^k} \equiv \begin{cases} 4C^2 \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

For any odd prime p and $m \in \mathbb{Z}_p$ we also establish the following general congruences:

$$(1.10) \quad \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \equiv P_{\frac{p-1}{2}} \left(\sqrt{1 - \frac{64}{m}} \right)^2 \pmod{p^2} \quad \text{for } m \not\equiv 0 \pmod{p},$$

$$(1.11) \quad \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 \pmod{p^2}.$$

2. Congruences for $P_{[\frac{p}{4}]}(t) \pmod{p}$.

Lemma 2.1. *Let p be an odd prime and $k \in \{0, 1, \dots, [\frac{p}{4}]\}$. Then*

$$\begin{aligned} \text{(i)} \quad & \binom{\frac{p-1}{2} - k}{k} \equiv \frac{1}{(-4)^k} \binom{4k}{2k} \pmod{p}, \\ \text{(ii)} \quad & \binom{[\frac{p}{4}] + k}{2k} \equiv \frac{1}{(-64)^k} \binom{4k}{2k} \pmod{p}, \\ \text{(iii)} \quad & \binom{p-1-2k}{\frac{p-1}{2}} \equiv \frac{(-1)^{\frac{p-1}{2}}}{16^k} \binom{4k}{2k} \pmod{p}. \end{aligned}$$

Proof. It is clear that

$$\frac{\binom{\frac{p-1}{2} - k}{k}}{\binom{\frac{p-1}{2}}{2k}} = \frac{(2k)!}{\frac{p-1}{2} \cdot \frac{p-3}{2} \cdots (\frac{p-1}{2} - k + 1) \cdot k!} \equiv \frac{(-2)^k \cdot (2k)!}{1 \cdot 3 \cdots (2k-1) \cdot k!} = (-4)^k \pmod{p}.$$

This together with (1.1) yields (i).

Suppose $r = 1$ or 3 according as $4 \mid p - 1$ or $4 \mid p - 3$. Then clearly

$$\begin{aligned} \binom{\left[\frac{p}{4}\right] + k}{2k} &= \frac{\left(\frac{p-r}{4} + k\right)\left(\frac{p-r}{4} + k - 1\right) \cdots \left(\frac{p-r}{4} - k + 1\right)}{(2k)!} \\ &\equiv (-1)^k \frac{(4k - r)(4k - r - 4) \cdots (4 - r) \cdot r(r + 4) \cdots (4k + r - 4)}{4^{2k} \cdot (2k)!} \\ &= \frac{(-1)^k \cdot (4k)!}{2^{2k} \cdot (2k)! \cdot 4^{2k} \cdot (2k)!} = \frac{\binom{4k}{2k}}{(-64)^k} \pmod{p} \end{aligned}$$

and

$$\begin{aligned} &(-1)^{\frac{p-1}{2}} \binom{p-1-2k}{\frac{p-1}{2}} \\ &= (-1)^{\frac{p-1}{2}} \frac{(p-1-2k)(p-2-2k) \cdots (p - (\frac{p-1}{2} + 2k))}{\frac{p-1}{2}!} \\ &\equiv \frac{(2k+1)(2k+2) \cdots (\frac{p-1}{2} + 2k)}{\frac{p-1}{2}!} = \frac{(\frac{p-1}{2} + 2k)(\frac{p-1}{2} + 2k - 1) \cdots (\frac{p-1}{2} + 1)}{(2k)!} \\ &\equiv \frac{(4k-1)(4k-3) \cdots 3 \cdot 1}{2^{2k} \cdot (2k)!} = \frac{(4k)!}{2^{2k} \cdot (2k)! \cdot 2^{2k} \cdot (2k)!} = \frac{1}{2^{4k}} \binom{4k}{2k} \pmod{p}. \end{aligned}$$

Thus (ii) and (iii) are true and the proof is complete.

Lemma 2.2. *Let p be an odd prime and let t be a variable. Then*

$$\begin{aligned} P_{\left[\frac{p}{4}\right]}(t) &\equiv \sum_{k=0}^{\left[\frac{p}{4}\right]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{1-t}{128}\right)^k \\ &\equiv (-1)^{\left[\frac{p}{4}\right]} \sum_{k=0}^{\left[\frac{p}{4}\right]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{1+t}{128}\right)^k \pmod{p}. \end{aligned}$$

Proof. It is known that ([G, (3.135)])

$$(2.1) \quad P_n(t) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(\frac{t-1}{2}\right)^k.$$

Observe that $\binom{n}{k} \binom{n+k}{k} = \binom{n+k}{2k} \binom{2k}{k}$. By (2.1) and Lemma 2.1(ii) we have

$$P_{\left[\frac{p}{4}\right]}(t) = \sum_{k=0}^{\left[\frac{p}{4}\right]} \binom{\left[\frac{p}{4}\right] + k}{2k} \binom{2k}{k} \left(\frac{t-1}{2}\right)^k \equiv \sum_{k=0}^{\left[\frac{p}{4}\right]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-64)^k} \left(\frac{t-1}{2}\right)^k \pmod{p}.$$

By (1.6), $P_{\left[\frac{p}{4}\right]}(t) = (-1)^{\left[\frac{p}{4}\right]} P_{\left[\frac{p}{4}\right]}(-t)$. Thus the result follows.

Theorem 2.1. *Let p be a prime greater than 3 and let t be a variable. Then*

$$\begin{aligned} P_{[\frac{p}{4}]}(t) &\equiv - \sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \\ &\equiv - \left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left(x^3 - \frac{3}{2}(3t+5)x + 9t+7\right)^{\frac{p-1}{2}} \pmod{p}. \end{aligned}$$

Proof. For any positive integer k it is well known that (see [IR, Lemma 2, p.235])

$$\sum_{x=0}^{p-1} x^k \equiv \begin{cases} p-1 \pmod{p} & \text{if } p-1 \mid k, \\ 0 \pmod{p} & \text{if } p-1 \nmid k. \end{cases}$$

Suppose that $(x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} = \sum_{k=1}^{3(p-1)/2} a_k x^k$. From the above we deduce that

$$a_{p-1} \equiv - \sum_{k=1}^{3(p-1)/2} a_k \sum_{x=0}^{p-1} x^k = - \sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \pmod{p}.$$

On the other hand,

$$\begin{aligned} &(x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \\ &= x^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} (x^2 + 4x)^{\frac{p-1}{2}-k} (2(1-t))^k \\ &= x^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} \sum_{r=0}^{\frac{p-1}{2}-k} \binom{\frac{p-1}{2}-k}{r} x^{2r} (4x)^{\frac{p-1}{2}-k-r} (2(1-t))^k. \end{aligned}$$

Hence, applying (1.1) and Lemmas 2.1-2.2 we obtain

$$\begin{aligned} a_{p-1} &= \sum_{k=0}^{[p/4]} \binom{\frac{p-1}{2}}{k} \binom{\frac{p-1}{2}-k}{k} 4^{\frac{p-1}{2}-2k} (2(1-t))^k \\ &\equiv \sum_{k=0}^{[p/4]} \frac{1}{(-4)^k} \binom{2k}{k} \frac{1}{(-4)^k} \binom{4k}{2k} 4^{-2k} \cdot 2^k (1-t)^k \\ &= \sum_{k=0}^{[p/4]} \binom{4k}{2k} \binom{2k}{k} \left(\frac{1-t}{128}\right)^k \equiv P_{[\frac{p}{4}]}(t) \pmod{p}. \end{aligned}$$

Therefore,

$$\begin{aligned}
P_{[\frac{p}{4}]}(t) &\equiv a_{p-1} \equiv - \sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \\
&\equiv - \sum_{x=0}^{p-1} \left(\left(x - \frac{4}{3}\right)^3 + 4\left(x - \frac{4}{3}\right)^2 + 2(1-t)\left(x - \frac{4}{3}\right) \right)^{\frac{p-1}{2}} \\
&= - \sum_{x=0}^{p-1} \left(x^3 - \frac{2}{3}(3t+5)x + \frac{8}{27}(9t+7) \right)^{\frac{p-1}{2}} \\
&\equiv - \sum_{x=0}^{p-1} \left(\left(\frac{2x}{3}\right)^3 - \frac{2}{3}(3t+5) \cdot \frac{2x}{3} + \frac{8}{27}(9t+7) \right)^{\frac{p-1}{2}} \\
&= - \left(\frac{8}{27}\right)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} \left(x^3 - \frac{3}{2}(3t+5)x + 9t+7 \right)^{\frac{p-1}{2}} \pmod{p}.
\end{aligned}$$

This yields the result.

Corollary 2.1. *Let $p > 3$ be a prime and $t \in \mathbb{Z}_p$. Then*

$$\sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(3t+5)}{2}x + 9t+7}{p} \right) = \left(\frac{2}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 + \frac{3(3t-5)}{2}x + 9t-7}{p} \right).$$

Proof. Since $P_{[\frac{p}{4}]}(t) = (-1)^{[\frac{p}{4}]} P_{[\frac{p}{4}]}(-t)$, by Theorem 2.1 we obtain

$$\begin{aligned}
&\sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(3t+5)}{2}x + 9t+7}{p} \right) \\
&\equiv (-1)^{[\frac{p}{4}]} \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(-3t+5)}{2}x - 9t+7}{p} \right) = \left(\frac{-2}{p}\right) \sum_{x=0}^{p-1} \left(\frac{(-x)^3 + \frac{3(3t-5)}{2}(-x) - 9t+7}{p} \right) \\
&= \left(\frac{2}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 + \frac{3(3t-5)}{2}x + 9t-7}{p} \right) \pmod{p}.
\end{aligned}$$

For $p \in \{5, 11, 13\}$ it is easy to check that the result is true for $t = 0, 1, \dots, p-1$. Now suppose $p \geq 17$. By Hasse's estimate ([C, Theorem 14.12, p.315]),

$$\left| \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(\pm 3t+5)}{2}x + 9t \pm 7}{p} \right) \right| \leq 2\sqrt{p}.$$

Since $4\sqrt{p} < p$, from the above we deduce the result for $p \geq 17$.

Theorem 2.2. *Let p be a prime greater than 3. Then*

$$\begin{aligned}
P_{[\frac{p}{4}]} \left(-\frac{7}{9} \right) &\equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{72^k} \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{576^k} \\
&\equiv \begin{cases} (-1)^{\frac{p-1}{4}} \left(\frac{p}{3}\right) 2a \pmod{p} & \text{if } 4 \mid p-1, p = a^2 + b^2 \text{ and } 4 \mid a-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}
\end{aligned}$$

Proof. From [BEW, Theorem 6.2.9] or [S1, (2.15)-(2.16)] we have

$$(2.2) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 - 4x}{p} \right) = \begin{cases} -2a & \text{if } p \equiv 1 \pmod{4}, p = a^2 + b^2 \text{ and } a \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Thus taking $t = -7/9$ in Lemma 2.2 and Theorem 2.1 we deduce the result.

Theorem 2.3. *Let p be a prime greater than 3. Then*

$$\begin{aligned} P_{[\frac{p}{4}]} \left(-\frac{5}{3} \right) &\equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{48^k} \equiv (-1)^{[p/4]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-192)^k} \\ &\equiv \begin{cases} 2A \pmod{p} & \text{if } 3 \mid p-1, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Proof. It is known that (see for example [S1, (2.7)-(2.9)] or [BEW, pp. 195-196])

$$(2.3) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 - 8}{p} \right) = \begin{cases} -2A \left(\frac{-2}{p} \right) & \text{if } 3 \mid p-1, p = A^2 + 3B^2 \text{ and } 3 \mid A-1, \\ 0 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Thus, putting $t = -5/3$ in Lemma 2.2 and Theorem 2.1 we deduce the result.

Theorem 2.4. *Let $p \neq 2, 3, 7$ be a prime. Then*

$$\begin{aligned} P_{[\frac{p}{4}]} \left(-\frac{65}{63} \right) &\equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{63^k} \equiv (-1)^{[p/4]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-4032)^k} \\ &\equiv \begin{cases} 2C \left(\frac{p}{3} \right) \left(\frac{C}{7} \right) \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

Proof. Putting $t = -65/63$ in Lemma 2.2 we get

$$P_{[\frac{p}{4}]} \left(-\frac{65}{63} \right) \equiv \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{63^k} \equiv (-1)^{[p/4]} \sum_{k=0}^{[p/4]} \frac{\binom{4k}{2k} \binom{2k}{k}}{(-4032)^k} \pmod{p}.$$

Taking $t = -65/63$ in Theorem 2.1 we see that

$$\begin{aligned} P_{[\frac{p}{4}]} \left(-\frac{65}{63} \right) &\equiv - \sum_{x=0}^{p-1} \left(\frac{x^3 + 4x^2 + \frac{256}{63}x}{p} \right) = - \sum_{x=0}^{p-1} \left(\frac{\left(\frac{4}{21}x \right)^3 + 4 \left(\frac{4}{21}x \right)^2 + \frac{256}{63} \cdot \frac{4}{21}x}{p} \right) \\ &= - \sum_{x=0}^{p-1} \left(\frac{\left(\frac{4}{21} \right)^3 (x^3 + 21x^2 + 112x)}{p} \right) \pmod{p}. \end{aligned}$$

From [R1,R2] we have

$$(2.4) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 + 21x^2 + 112x}{p} \right) = \begin{cases} -2C \left(\frac{C}{7} \right) & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Thus the result follows.

For related conjectures concerning Theorems 2.2-2.4, see [Su1, Conjectures A47-A49].

Lemma 2.3. *Let p be an odd prime and let u be a variable. Then*

$$(\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) \equiv (-1)^{\lfloor \frac{p}{4} \rfloor} \sum_{k=0}^{\lfloor \frac{p}{4} \rfloor} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} u^{\frac{p-1}{2}-k} \pmod{p}.$$

Proof. From (1.5) we see that

$$(\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) = \frac{1}{2^{\frac{p-1}{2}}} \sum_{k=0}^{\lfloor \frac{p}{4} \rfloor} \binom{\frac{p-1}{2}}{k} \binom{p-1-2k}{\frac{p-1}{2}} (-1)^k u^{\frac{p-1}{2}-k}.$$

Thus applying (1.1) and Lemma 2.1 we obtain

$$(\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{(-4)^k} \binom{2k}{k} \cdot \frac{1}{2^{4k}} \binom{4k}{2k} (-1)^k u^{\frac{p-1}{2}-k} \pmod{p}.$$

Noting that $\left(\frac{-2}{p}\right) = (-1)^{\lfloor \frac{p}{4} \rfloor}$ we then obtain the result.

Theorem 2.5. *Let $p > 3$ be a prime, $u \in \mathbb{Z}_p$ and $u \not\equiv 0 \pmod{p}$. Then*

$$\begin{aligned} (\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) &\equiv \left(\frac{u}{p}\right) P_{\lfloor \frac{p}{4} \rfloor} \left(\frac{2}{u} - 1\right) \equiv - \sum_{x=0}^{p-1} \left(\frac{x^3 - 2ux^2 + ux}{p}\right) \\ &\equiv - \left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{ux^3 - (3u+9)x + 18 - 2u}{p}\right) \pmod{p}. \end{aligned}$$

Proof. By (1.6), Lemmas 2.2 and 2.3 we have

$$\begin{aligned} P_{\lfloor \frac{p}{4} \rfloor} \left(\frac{2}{u} - 1\right) &= \left(\frac{-2}{p}\right) P_{\lfloor \frac{p}{4} \rfloor} \left(1 - \frac{2}{u}\right) \equiv (-1)^{\lfloor \frac{p}{4} \rfloor} \sum_{k=0}^{\lfloor \frac{p}{4} \rfloor} \binom{4k}{2k} \binom{2k}{k} \frac{1}{(64u)^k} \\ &\equiv \left(\frac{u}{p}\right) (\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) \pmod{p}. \end{aligned}$$

By Theorem 2.1, we have

$$\begin{aligned} &P_{\lfloor \frac{p}{4} \rfloor} (1 - 2/u) \\ &\equiv - \sum_{x=0}^{p-1} \left(\frac{x^3 + 4x^2 + \frac{4}{u}x}{p}\right) = - \sum_{x=0}^{p-1} \left(\frac{(-\frac{2x}{u})^3 + 4(-\frac{2x}{u})^2 + \frac{4}{u}(-\frac{2x}{u})}{p}\right) \\ &= - \sum_{x=0}^{p-1} \left(\frac{(-\frac{2}{u})^3 (x^3 - 2ux^2 + ux)}{p}\right) = - \left(\frac{-2u}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 2ux^2 + ux}{p}\right) \pmod{p} \end{aligned}$$

and

$$\begin{aligned} P_{\lfloor \frac{p}{4} \rfloor} \left(\frac{2}{u} - 1\right) &\equiv - \left(\frac{6}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3(u+3)}{u}x + \frac{18-2u}{u}}{p}\right) \\ &= - \left(\frac{6u}{p}\right) \sum_{x=0}^{p-1} \left(\frac{ux^3 - 3(u+3)x + 18 - 2u}{p}\right) \pmod{p}. \end{aligned}$$

Thus the theorem is proved.

3. Congruences for $\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{m^k} \binom{2k}{k}^3 \pmod{p^2}$.

Lemma 3.1. *For any nonnegative integers m and n we have*

$$\sum_{k=0}^m \binom{2k}{k}^2 \binom{n+k}{2k} \binom{k}{m-k} = \sum_{k=0}^m \binom{2k}{k} \binom{n+k}{2k} \binom{2m-2k}{m-k} \binom{n+m-k}{2m-2k}.$$

Proof. Let m and n be nonnegative integers. For $k \in \{0, 1, \dots, m\}$ set

$$F_1(m, k) = \binom{2k}{k} \binom{n+k}{2k} \binom{2k}{k} \binom{k}{m-k},$$

$$F_2(m, k) = \binom{2k}{k} \binom{n+k}{2k} \binom{2(m-k)}{m-k} \binom{n+m-k}{2(m-k)}.$$

For $k \in \{0, 1, \dots, m+1\}$ set

$$G_1(m, k) = -(m+2)k^2 \binom{2k}{k}^2 \binom{n+k}{2k} \binom{k}{m+2-k},$$

$$G_2(m, k) = -\frac{k^2(3mn^2 - 2n^2k + m^2 + 3mn - mk + 6n^2 - 2nk + 4m + 6n - 2k + 4)}{(m+2-k)^2}$$

$$\times \binom{2k}{k} \binom{n+k}{2k} \binom{2(m+1-k)}{m+1-k} \binom{n+m+1-k}{2(m+1-k)}.$$

For $i = 1, 2$ and $k \in \{0, 1, \dots, m\}$, using Maple it is easy to check that

$$(3.1) \quad (m+2)^3 F_i(m+2, k) + (2m+3)(m^2 - 2n^2 + 3m - 2n + 2) F_i(m+1, k) \\ + (m+1)(m+2n+2)(m-2n) F_i(m, k) = G_i(m, k+1) - G_i(m, k).$$

Set $S_i(r) = \sum_{k=0}^r F_i(r, k)$ for $r = 0, 1, 2, \dots$. Then

$$(m+2)^3 (S_i(m+2) - F_i(m+2, m+2) - F_i(m+2, m+1)) \\ + (2m+3)(m^2 - 2n^2 + 3m - 2n + 2) (S_i(m+1) - F_i(m+1, m+1)) \\ + (m+1)(m+2n+2)(m-2n) S_i(m) \\ = (m+2)^3 \sum_{k=0}^m F_i(m+2, k) + (2m+3)(m^2 - 2n^2 + 3m - 2n + 2) \sum_{k=0}^m F_i(m+1, k) \\ + (m+1)(m+2n+2)(m-2n) \sum_{k=0}^m F_i(m, k) \\ = \sum_{k=0}^m (G_i(m, k+1) - G_i(m, k)) = G_i(m, m+1) - G_i(m, 0) = G_i(m, m+1).$$

From the above we deduce that for $i = 1, 2$ and $m = 0, 1, 2, \dots$,

$$(3.2) \quad (m+2)^3 S_i(m+2) + (2m+3)(m^2 - 2n^2 + 3m - 2n + 2) S_i(m+1) \\ + (m+1)(m+2n+2)(m-2n) S_i(m) \\ = G_i(m, m+1) + (m+2)^3 (F_i(m+2, m+2) + F_i(m+2, m+1)) \\ + (2m+3)(m^2 - 2n^2 + 3m - 2n + 2) F_i(m+1, m+1) = 0.$$

Since $S_1(0) = 1 = S_2(0)$ and $S_1(1) = 2n(n+1) = S_2(1)$, from (3.2) we deduce $S_1(r) = S_2(r)$ for all $r = 0, 1, 2, \dots$. This completes the proof.

Remark 3.1 We actually find (3.1) and prove Lemma 3.1 by using WZ method and Maple. For the WZ method, see [PWZ].

Definition 3.1. Note that $\binom{n}{k} \binom{n+k}{k} = \binom{2k}{k} \binom{n+k}{2k}$. For any nonnegative integer n we define

$$S_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} x^k = \sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} x^k.$$

Lemma 3.2. Let n be a nonnegative integer. Then

$$S_n(x) = P_n(\sqrt{1+4x})^2 \quad \text{and} \quad P_n(x)^2 = S_n\left(\frac{x^2-1}{4}\right).$$

Proof. From (2.1) we have

$$\begin{aligned} P_n(x)^2 &= \left(\sum_{k=0}^n \binom{2k}{k} \binom{n+k}{2k} \left(\frac{x-1}{2}\right)^k \right) \left(\sum_{r=0}^n \binom{2r}{r} \binom{n+r}{2r} \left(\frac{x-1}{2}\right)^r \right) \\ &= \sum_{m=0}^{2n} \left(\frac{x-1}{2}\right)^m \sum_{k=0}^m \binom{2k}{k} \binom{n+k}{2k} \binom{2m-2k}{m-k} \binom{n+m-k}{2m-2k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} S_n\left(\frac{x^2-1}{4}\right) &= \sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(\frac{x-1}{2}\right)^k \left(1 + \frac{x-1}{2}\right)^k \\ &= \sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \left(\frac{x-1}{2}\right)^k \sum_{r=0}^k \binom{k}{r} \left(\frac{x-1}{2}\right)^r \\ &= \sum_{m=0}^{2n} \left(\frac{x-1}{2}\right)^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{n+k}{2k} \binom{k}{m-k}. \end{aligned}$$

Hence, from the above and Lemma 3.1 we deduce

$$P_n(x)^2 = S_n\left(\frac{x^2-1}{4}\right) \quad \text{and so} \quad P_n(\sqrt{1+4x})^2 = S_n(x).$$

This proves the lemma.

Remark 3.2 Since $S_n(-\frac{1}{4}) = P_n(0)^2$ by Lemma 3.2, using (1.6) we deduce that

$$\sum_{k=0}^n \binom{2k}{k}^2 \binom{n+k}{2k} \frac{1}{(-4)^k} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{1}{2^{2n}} \binom{n}{n/2}^2 & \text{if } n \text{ is even.} \end{cases}$$

This is an identity due to Bell ([G, (6.35)]).

Theorem 3.1. *Let p be an odd prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \equiv P_{\frac{p-1}{2}} \left(\sqrt{1 - \frac{64}{m}} \right)^2 \pmod{p^2}.$$

Proof. By Definition 3.1 and (1.1) we have

$$S_{\frac{p-1}{2}} \left(-\frac{16}{m} \right) = \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^2 \binom{\frac{p-1}{2} + k}{2k} \left(-\frac{16}{m} \right)^k \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \pmod{p^2}.$$

On the other hand, by Lemma 3.2 we get

$$S_{\frac{p-1}{2}} \left(-\frac{16}{m} \right) = P_{\frac{p-1}{2}} \left(\sqrt{1 + 4 \left(-\frac{16}{m} \right)} \right)^2 = P_{\frac{p-1}{2}} \left(\sqrt{\frac{m-64}{m}} \right)^2.$$

Thus the result follows.

Theorem 3.2. *Let p be an odd prime, $m \in \mathbb{Z}_p$ and $m \not\equiv 0, 64 \pmod{p}$. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \equiv \left(\frac{m(m-64)}{p} \right) P_{[\frac{p}{4}]} \left(\frac{m+64}{m-64} \right)^2 \pmod{p}.$$

Moreover,

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \equiv 0 \pmod{p^2} \iff P_{[\frac{p}{4}]} \left(\frac{m+64}{m-64} \right) \equiv 0 \pmod{p}.$$

Proof. Set $u = 1 - 64/m$. From Theorem 2.5 we know that

$$(\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) \equiv \left(\frac{u}{p} \right) P_{[\frac{p}{4}]} \left(\frac{2}{u} - 1 \right) \pmod{p}.$$

Thus,

$$\left(\frac{u}{p} \right) P_{\frac{p-1}{2}}(\sqrt{u})^2 \equiv u^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u})^2 \equiv P_{[\frac{p}{4}]} \left(\frac{2}{u} - 1 \right)^2 \pmod{p}.$$

It then follows from Theorem 3.1 that

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} \equiv P_{\frac{p-1}{2}}(\sqrt{u})^2 \equiv \left(\frac{u}{p} \right) P_{[\frac{p}{4}]} \left(\frac{2}{u} - 1 \right)^2 = \left(\frac{m(m-64)}{p} \right) P_{[\frac{p}{4}]} \left(\frac{m+64}{m-64} \right)^2 \pmod{p}.$$

From (1.5) we see that $(\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) \in \mathbb{Z}_p$. Thus, by Theorems 3.1 and 2.5,

$$\begin{aligned} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{m^k} &\equiv 0 \pmod{p^2} \\ \iff P_{\frac{p-1}{2}}(\sqrt{u})^2 &\equiv 0 \pmod{p^2} \iff ((\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}))^2 \equiv 0 \pmod{p^2} \\ \iff (\sqrt{u})^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(\sqrt{u}) &\equiv 0 \pmod{p} \iff P_{[\frac{p}{4}]} \left(\frac{2}{u} - 1 \right) \equiv 0 \pmod{p} \\ \iff P_{[\frac{p}{4}]} \left(\frac{m+64}{m-64} \right) &\equiv 0 \pmod{p}. \end{aligned}$$

Thus the theorem is proved.

Theorem 3.3. *Let p be an odd prime. Then*

$$\begin{aligned}
\text{(i)} \quad & \sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{4096^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3, 5, 6 \pmod{7}, \\
\text{(ii)} \quad & \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-512)^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 3 \pmod{4}, \\
\text{(iii)} \quad & \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{16^k} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{256^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 2 \pmod{3}, \\
\text{(iv)} \quad & \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2} \quad \text{for } p \equiv 5, 7 \pmod{8}.
\end{aligned}$$

Proof. Taking $m = 1, 4096, -8, -512, 16, 256, -64$ in Theorem 3.2 and then applying Theorems 2.2-2.4 and (1.6) we obtain the result.

Theorem 3.4. *Let p be an odd prime.*

(i) *If $p \equiv 1, 2, 4 \pmod{7}$ and so $p = C^2 + 7D^2$ with $C, D \in \mathbb{Z}$, then*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}^3 \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{4096^k} \equiv 4C^2 - 2p \pmod{p^2}.$$

(ii) *If $p \equiv 1 \pmod{4}$ and so $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $2 \nmid a$, then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv (-1)^{\frac{p-1}{4}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-512)^k} \equiv 4a^2 - 2p \pmod{p^2}.$$

(iii) *If $p \equiv 1 \pmod{3}$ and so $p = A^2 + 3B^2$ with $A, B \in \mathbb{Z}$, then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{16^k} \equiv (-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{256^k} \equiv 4A^2 - 2p \pmod{p^2}.$$

(iv) *If $p \equiv 1, 3 \pmod{8}$ and so $p = c^2 + 2d^2$ with $c, d \in \mathbb{Z}$, then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv (-1)^{\frac{p-1}{2}} (4c^2 - 2p) \pmod{p^2}.$$

Proof. From [CH, (48), Tables II and III] we know that

$$P_{\frac{p-1}{2}}(3\sqrt{-7}) \equiv x - y\sqrt{-7} \pmod{p^2} \text{ for } p = x^2 + 7y^2 \text{ with } 4 \mid x + y - 1,$$

$$P_{\frac{p-1}{2}}\left(\frac{3\sqrt{7}}{8}\right) \equiv \left(\frac{\sqrt{-7}}{\sqrt{7}}\right)^{\frac{p-1}{2}} (x - y\sqrt{-7}) \pmod{p^2} \text{ for } p = x^2 + 7y^2 \text{ with } 4 \mid x + y - 1,$$

$$P_{\frac{p-1}{2}}(3) \equiv (-1)^{\frac{p-1}{4}} (x - y\sqrt{-1}) \pmod{p^2} \text{ for } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ with } 4 \mid x - 1,$$

$$P_{\frac{p-1}{2}}\left(\frac{3\sqrt{2}}{4}\right) \equiv (\sqrt{-1})^{\frac{y}{2}} (x - y\sqrt{-1}) \pmod{p^2} \text{ for } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ with } 4 \mid x - 1,$$

$$P_{\frac{p-1}{2}}(\sqrt{-3}) \equiv (-1)^{\frac{p-1}{2}} (x - y\sqrt{-3}) \pmod{p^2} \text{ for } p = x^2 + 3y^2 \equiv 1 \pmod{3} \text{ with } 4 \mid x + y - 1,$$

$$P_{\frac{p-1}{2}}\left(\frac{\sqrt{3}}{2}\right) \equiv \left(-\frac{\sqrt{-3}}{\sqrt{3}}\right)^{\frac{p-1}{2}} (x - y\sqrt{-3}) \pmod{p^2}$$

for $p = x^2 + 3y^2 \equiv 1 \pmod{3}$ with $4 \mid x + y - 1$,

$$P_{\frac{p-1}{2}}(\sqrt{2}) \equiv \left(\frac{\sqrt{-2}}{\sqrt{2}}\right)^{-y} (x - y\sqrt{-2}) \pmod{p^2}$$

for $p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}$ with $4 \mid x - 1$,

where $\sqrt{-d} \in \mathbb{Q}_p$ and $|x - y\sqrt{-d}|_p = 1$ ($|\cdot|_p$ is the usual valuation on \mathbb{Q}_p).

If $\left(\frac{-d}{p}\right) = 1$, $p = x^2 + dy^2$ and $A \in \mathbb{Q}_p$ with $A \equiv x - y\sqrt{-d} \pmod{p^2}$ and $|x - y\sqrt{-d}|_p = 1$, then

$$(A - x)^2 \equiv -dy^2 = x^2 - p \pmod{p^2} \quad \text{and so} \quad A^2 - 2xA \equiv -p \pmod{p^2}.$$

Hence $A \equiv 2x \pmod{p}$ and so $2xA \equiv (A - 2x)^2 + 2xA = A^2 - 2xA + 4x^2 \equiv 4x^2 - p \pmod{p^2}$. Therefore,

$$A \equiv 2x - \frac{p}{2x} \pmod{p^2} \quad \text{and so} \quad A^2 \equiv 4x^2 - 2p \pmod{p^2}.$$

From all the above we deduce that

$$P_{\frac{p-1}{2}}(3\sqrt{-7})^2 \equiv (-1)^{\frac{p-1}{2}} P_{\frac{p-1}{2}}\left(\frac{3\sqrt{7}}{8}\right)^2 \equiv 4x^2 - 2p \pmod{p^2} \text{ for } p = x^2 + 7y^2,$$

$$P_{\frac{p-1}{2}}(3)^2 \equiv (-1)^{\frac{p-1}{4}} P_{\frac{p-1}{2}}\left(\frac{3\sqrt{2}}{4}\right)^2 \equiv 4x^2 - 2p \pmod{p^2}$$

for $p = x^2 + y^2 \equiv 1 \pmod{4}$ with $2 \nmid x$ (see also [S2, Corollary 2.3 and Theorem 2.9]),

$$P_{\frac{p-1}{2}}(\sqrt{-3})^2 \equiv (-1)^{\frac{p-1}{2}} P_{\frac{p-1}{2}}\left(\frac{\sqrt{3}}{2}\right)^2 \equiv 4x^2 - 2p \pmod{p^2}$$

for $p = x^2 + 3y^2 \equiv 1 \pmod{3}$,

$$P_{\frac{p-1}{2}}(\sqrt{2})^2 \equiv (-1)^y (4x^2 - 2p) = (-1)^{\frac{p-1}{2}} (4x^2 - 2p) \pmod{p^2}$$

for $p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}$.

Now taking $m = 1, 4096, -8, -512, 16, 256, -64$ in Theorem 3.1 and then applying the above we derive the result.

Corollary 3.1. *Let p be a prime such that $p \neq 2, 3, 7$. Then*

$$\begin{aligned} & \sum_{k=0}^{(p-1)/2} k \binom{2k}{k}^3 \\ & \equiv \begin{cases} \frac{8}{21}(3p - 4C^2) \pmod{p^2} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ \frac{8}{21}p \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases} \end{aligned}$$

Proof. By [Su2, Theorem 1.3] we have

$$\sum_{k=0}^{(p-1)/2} (21k + 8) \binom{2k}{k}^3 \equiv \sum_{k=0}^{p-1} (21k + 8) \binom{2k}{k}^3 \equiv 8p \pmod{p^2}.$$

Thus applying Theorems 3.3(i) and 3.4(i) we conclude the result.

Remark 3.3 Theorems 3.3, 3.4 and Corollary 3.1 were conjectured by Zhi-Wei Sun in [Su1, Su3].

Lemma 3.3. *For any positive integer n we have the following identities:*

$$\begin{aligned} \sum_{k=1}^n \binom{2k}{k}^2 \binom{n+k}{2k} \frac{k}{(-4)^k} &= \begin{cases} -\frac{n^2}{2^{2n-2}} \left(\frac{n-1}{2}\right)^2 & \text{if } 2 \nmid n, \\ \frac{n(n+1)}{2^{2n}} \left(\frac{n}{2}\right)^2 & \text{if } 2 \mid n, \end{cases} \\ \sum_{k=1}^n \binom{2k}{k}^2 \binom{n+k}{2k} \frac{k^2}{(-4)^k} &= \begin{cases} -\frac{n^2(2n^2+2n-1)}{3 \cdot 2^{2n-2}} \left(\frac{n-1}{2}\right)^2 & \text{if } 2 \nmid n, \\ \frac{n^2(n+1)^2}{3 \cdot 2^{2n-1}} \left(\frac{n}{2}\right)^2 & \text{if } 2 \mid n, \end{cases} \\ \sum_{k=1}^n \binom{2k}{k}^2 \binom{n+k}{2k} \frac{k^3}{(-4)^k} &= \begin{cases} -\frac{n^2(4n^2(n+1)^2 - n(n+1)+1)}{15 \cdot 2^{2n-2}} \left(\frac{n-1}{2}\right)^2 & \text{if } 2 \nmid n, \\ \frac{n^2(n+1)^2(2n+1)^2}{15 \cdot 2^{2n}} \left(\frac{n}{2}\right)^2 & \text{if } 2 \mid n. \end{cases} \end{aligned}$$

Proof. We only prove the first identity. The other identities can be proved similarly. Let $f'(x) = \frac{d}{dx} f(x)$ be the derivative of $f(x)$. By Lemma 3.2, $S_n(x) = P_n(\sqrt{1+4x})^2$. If $2 \mid n$, then $P_n(\sqrt{1+4x})$ is a polynomial in $1+4x$. Thus

$$S'_n(x) = 2P_n(\sqrt{1+4x}) \cdot \frac{d}{dx} P_n(\sqrt{1+4x})$$

and so

$$S'_n\left(-\frac{1}{4}\right) = 2P_n(0) \frac{d}{dx} P_n(\sqrt{1+4x}) \Big|_{x=-\frac{1}{4}}.$$

From (1.5) we see that

$$\begin{aligned} \frac{d}{dx} P_n(\sqrt{1+4x}) &= \frac{d}{dx} \frac{1}{2^n} \sum_{k=0}^{n/2} \binom{n}{k} (-1)^k \binom{2n-2k}{n} (1+4x)^{\frac{n}{2}-k} \\ &= \frac{4}{2^n} \sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} (-1)^k \binom{2n-2k}{n} \left(\frac{n}{2} - k\right) (1+4x)^{\frac{n}{2}-k-1} \end{aligned}$$

and therefore

$$\begin{aligned} \frac{d}{dx} P_n(\sqrt{1+4x}) \Big|_{x=-\frac{1}{4}} &= \frac{4}{2^n} \binom{n}{\frac{n}{2}-1} (-1)^{\frac{n}{2}-1} \binom{2n-2(\frac{n}{2}-1)}{n} \\ &= (-1)^{\frac{n}{2}-1} \frac{n(n+1)}{2^{n-1}} \binom{n}{n/2}. \end{aligned}$$

Combining the above with (1.6) we get

$$S'_n\left(-\frac{1}{4}\right) = 2 \cdot \frac{(-1)^{n/2}}{2^n} \binom{n}{n/2} \cdot (-1)^{\frac{n}{2}-1} \frac{n(n+1)}{2^{n-1}} \binom{n}{n/2} = -\frac{n(n+1)}{2^{2n-2}} \binom{n}{n/2}^2.$$

By Definition 3.1,

$$(3.3) \quad S'_n\left(-\frac{1}{4}\right) = \sum_{k=1}^n \binom{2k}{k}^2 \binom{n+k}{2k} \frac{k}{(-4)^{k-1}}.$$

Thus the first identity is true when n is even.

Now we assume $2 \nmid n$. Set

$$A_n(x) = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} (-1)^k \binom{2n-2k}{n} (1+4x)^{\frac{n-1}{2}-k}.$$

From Lemma 3.2 and (1.5) we have

$$S_n(x) = (1+4x) \left(\frac{P_n(\sqrt{1+4x})}{\sqrt{1+4x}} \right)^2 = \frac{1}{2^{2n}} (1+4x) A_n(x)^2.$$

Thus,

$$S'_n(x) = \frac{1}{2^{2n}} (4A_n(x)^2 + (1+4x) \cdot 2A_n(x)A'_n(x))$$

and therefore

$$\begin{aligned} S'_n\left(-\frac{1}{4}\right) &= \frac{4}{2^{2n}} A_n\left(-\frac{1}{4}\right)^2 = \frac{4}{2^{2n}} \left(\binom{n}{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}} \binom{2n-(n-1)}{n} \right)^2 \\ &= \frac{4(n+1)^2}{2^{2n}} \binom{n}{\frac{n-1}{2}}^2 = \frac{4n^2}{2^{2n-2}} \binom{n-1}{\frac{n-1}{2}}^2. \end{aligned}$$

This together with (3.3) yields the first identity in the case $2 \nmid n$. The proof is now complete.

We remark that Lemma 3.3 can also be proved by using WZ method.

Lemma 3.4. *Let p be a prime of the form $4k+1$ and so $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $2 \nmid a$. Then*

$$\frac{1}{2^{p-1}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 \equiv 4a^2 - 2p \pmod{p^2}.$$

Proof. From [CDE] or [BEW] we know that for $a \equiv 1 \pmod{4}$,

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \frac{2^{p-1} + 1}{2} \left(2a - \frac{p}{2a}\right) \pmod{p^2}.$$

Set $q_p(2) = (2^{p-1} - 1)/p$. We then have

$$\begin{aligned} \frac{1}{2^{p-1}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 &\equiv \frac{1}{2^{p-1}} \cdot \frac{(2^{p-1} + 1)^2}{4} (4a^2 - 2p) \\ &= \frac{1}{4} \cdot \frac{(2 + pq_p(2))^2}{1 + pq_p(2)} (4a^2 - 2p) \equiv 4a^2 - 2p \pmod{p^2}. \end{aligned}$$

This proves the lemma.

Theorem 3.5. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} \sum_{k=1}^{\frac{p-1}{2}} \frac{k \binom{2k}{k}^3}{64^k} &\equiv \begin{cases} (2p - 2 + 2^{p-1}) \binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ \frac{p}{2} - a^2 \pmod{p^2} & \text{if } p = a^2 + 4b^2 \equiv 1 \pmod{4}, \end{cases} \\ \sum_{k=1}^{\frac{p-1}{2}} \frac{k^2 \binom{2k}{k}^3}{64^k} &\equiv \begin{cases} (1 - p - 2^{p-2}) \binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ \frac{1}{12} (2a^2 - p) \pmod{p^2} & \text{if } p = a^2 + 4b^2 \equiv 1 \pmod{4}, \end{cases} \\ \sum_{k=1}^{\frac{p-1}{2}} \frac{k^3 \binom{2k}{k}^3}{64^k} &\equiv \begin{cases} \frac{1}{5} (2^{p-2} - 1 + p) \binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}. \end{cases} \end{aligned}$$

Proof. By (1.1) and Lemma 3.3,

$$\begin{aligned} \sum_{k=1}^{\frac{p-1}{2}} \frac{k \binom{2k}{k}^3}{64^k} &\equiv \sum_{k=1}^{\frac{p-1}{2}} \binom{2k}{k}^2 \binom{\frac{p-1}{2} + k}{2k} \frac{k}{(-4)^k} \\ &= \begin{cases} -\frac{(p-1)^2}{2^{p-1}} \binom{(p-3)/2}{(p-3)/4}^2 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ \frac{p-1}{2} \cdot \frac{p+1}{2^{p-1}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 \equiv -\frac{1}{4} \cdot \frac{1}{2^{p-1}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}. \end{cases} \end{aligned}$$

Set $q_p(2) = (2^{p-1} - 1)/p$. For $p \equiv 3 \pmod{4}$ we see that

$$\begin{aligned} -\frac{(p-1)^2}{2^{p-1}} &\equiv \frac{2p-1}{1 + pq_p(2)} \equiv (2p-1)(1 - pq_p(2)) \\ &\equiv -1 + (2 + q_p(2))p = 2p - 2 + 2^{p-1} \pmod{p^2}. \end{aligned}$$

Now combining all the above with Lemma 3.4 we obtain the first congruence. The remaining congruences can be proved similarly.

Remark 3.4 In [Su3,(1.10)], Zhi-Wei Sun obtained the congruence for $\sum_{k=1}^{\frac{p-1}{2}} \frac{k^3 \binom{2k}{k}^3}{64^k} \pmod{p}$.

He also conjectured that $\sum_{k=1}^{\frac{p-1}{2}} \frac{k^3 \binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^2}$ for any prime $p \equiv 1 \pmod{4}$ with $p \neq 5$.

4. A general congruence modulo p^2 .

Lemma 4.1. *For any nonnegative integer n , we have*

$$\sum_{k=0}^n \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{n-k} (-64)^{n-k} = \sum_{k=0}^n \binom{2k}{k} \binom{4k}{2k} \binom{2(n-k)}{n-k} \binom{4(n-k)}{2(n-k)}.$$

Proof. Let m be a nonnegative integer. For $k \in \{0, 1, \dots, m\}$ set

$$F_1(m, k) = \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k},$$

$$F_2(m, k) = \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)}.$$

For $k \in \{0, 1, \dots, m+1\}$ set

$$G_1(m, k) = 64k^2(m+2) \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m+2-k} (-64)^{m+1-k},$$

$$G_2(m, k) = \frac{4k^2(16m^2 - 16mk + 55m - 26k + 46)}{(m+2-k)^2}$$

$$\times \binom{2k}{k} \binom{4k}{2k} \binom{2(m+1-k)}{m+1-k} \binom{4(m+1-k)}{2(m+1-k)}.$$

For $i = 1, 2$ and $k \in \{0, 1, \dots, m\}$, using Maple it is easy to check that

$$(4.1) \quad \begin{aligned} & (m+2)^3 F_i(m+2, k) - 8(2m+3)(8m^2 + 24m + 19) F_i(m+1, k) \\ & + 1024(m+1)(2m+1)(2m+3) F_i(m, k) = G_i(m, k+1) - G_i(m, k). \end{aligned}$$

Set $S_i(n) = \sum_{k=0}^n F_i(n, k)$ for $n = 0, 1, 2, \dots$. Then

$$\begin{aligned} & (m+2)^3 (S_i(m+2) - F_i(m+2, m+2) - F_i(m+2, m+1)) \\ & - 8(2m+3)(8m^2 + 24m + 19) (S_i(m+1) - F_i(m+1, m+1)) \\ & + 1024(m+1)(2m+1)(2m+3) S_i(m) \\ & = (m+2)^3 \sum_{k=0}^m F_i(m+2, k) - 8(2m+3)(8m^2 + 24m + 19) \sum_{k=0}^m F_i(m+1, k) \\ & + 1024(m+1)(2m+1)(2m+3) \sum_{k=0}^m F_i(m, k) \\ & = \sum_{k=0}^m (G_i(m, k+1) - G_i(m, k)) = G_i(m, m+1) - G_i(m, 0) = G_i(m, m+1). \end{aligned}$$

From the above we deduce that for $i = 1, 2$ and $m = 0, 1, 2, \dots$,

$$(4.2) \quad \begin{aligned} & (m+2)^3 S_i(m+2) - 8(2m+3)(8m^2 + 24m + 19) S_i(m+1) \\ & + 1024(m+1)(2m+1)(2m+3) S_i(m) \\ & = G_i(m, m+1) + (m+2)^3 (F_i(m+2, m+2) + F_i(m+2, m+1)) \\ & - 8(2m+3)(8m^2 + 24m + 19) F_i(m+1, m+1) = 0. \end{aligned}$$

Since $S_1(0) = 1 = S_2(0)$ and $S_1(1) = 24 = S_2(1)$, from (4.2) we deduce $S_1(n) = S_2(n)$ for all $n = 0, 1, 2, \dots$. This completes the proof.

Theorem 4.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} x^k \sum_{r=0}^k \binom{k}{r} (-64x)^r \\ &= \sum_{m=0}^{2(p-1)} x^m \sum_{k=0}^{\min\{m, p-1\}} \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k}. \end{aligned}$$

Suppose $p \leq m \leq 2p-2$ and $0 \leq k \leq p-1$. If $k > \frac{p}{2}$, then $p \mid \binom{2k}{k}$ and so $p^2 \mid \binom{2k}{k}^2$. If $k < \frac{p}{2}$, then $m-k \geq p-k > k$ and so $\binom{k}{m-k} = 0$. Thus, from the above and Lemma 4.1 we deduce that

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} (x(1-64x))^k \\ &\equiv \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k}^2 \binom{4k}{2k} \binom{k}{m-k} (-64)^{m-k} \\ &= \sum_{m=0}^{p-1} x^m \sum_{k=0}^m \binom{2k}{k} \binom{4k}{2k} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{m=k}^{p-1} \binom{2(m-k)}{m-k} \binom{4(m-k)}{2(m-k)} x^{m-k} \\ &= \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=0}^{p-1-k} \binom{2r}{r} \binom{4r}{2r} x^r \\ &= \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \right)^2 - \sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \pmod{p^2}. \end{aligned}$$

Now suppose $0 \leq k \leq p-1$ and $p-k \leq r \leq p-1$. If $k \geq \frac{3p}{4}$, then $p^2 \nmid (2k)!$, $p^3 \mid (4k)!$ and so $\binom{2k}{k} \binom{4k}{2k} = \frac{(4k)!}{(2k)!k!^2} \equiv 0 \pmod{p^2}$. If $k < \frac{p}{4}$, then $r \geq p-k > \frac{3p}{4}$ and so $\binom{2r}{r} \binom{4r}{2r} = \frac{(4r)!}{(2r)!r!^2} \equiv 0 \pmod{p^2}$. If $\frac{p}{4} < k < \frac{p}{2}$, then $r \geq p-k > \frac{p}{2}$, $p \mid \binom{2r}{r}$ and $p \mid \binom{4k}{2k}$. If $\frac{p}{2} < k < \frac{3p}{4}$, then $r \geq p-k > \frac{p}{4}$, $p \mid \binom{2k}{k}$ and $p \mid \binom{2r}{r} \binom{4r}{2r}$. Hence we always have $\binom{2k}{k} \binom{4k}{2k} \binom{2r}{r} \binom{4r}{2r} \equiv 0 \pmod{p^2}$ and so

$$\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} x^k \sum_{r=p-k}^{p-1} \binom{2r}{r} \binom{4r}{2r} x^r \equiv 0 \pmod{p^2}.$$

Now combining all the above we obtain the result.

Corollary 4.1. *Let p be an odd prime and $m \in \mathbb{Z}_p$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv \left(\sum_{k=0}^{p-1} \binom{2k}{k} \binom{4k}{2k} \left(\frac{1 - \sqrt{1 - 256/m}}{128} \right)^k \right)^2 \pmod{p^2}.$$

Proof. Taking $x = \frac{1 - \sqrt{1 - 256/m}}{128}$ in Theorem 4.1 we deduce the result.

Theorem 4.2. *Let p be an odd prime, $m \in \mathbb{Z}_p$, $m \not\equiv 0 \pmod{p}$ and $t = \sqrt{1 - 256/m}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv P_{[\frac{p}{4}]}(t)^2 \equiv \left(\sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \right)^2 \pmod{p}.$$

Moreover, if $P_{[\frac{p}{4}]}(t) \equiv 0 \pmod{p}$ or $\sum_{x=0}^{p-1} (x^3 + 4x^2 + 2(1-t)x)^{\frac{p-1}{2}} \equiv 0 \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{m^k} \equiv 0 \pmod{p^2}.$$

Proof. For $\frac{p}{2} < k < p$, $\binom{2k}{k} = \frac{(2k)!}{k!^2} \equiv 0 \pmod{p}$. For $\frac{p}{4} < k < \frac{p}{2}$, $\binom{4k}{2k} = \frac{(4k)!}{(2k)!^2} \equiv 0 \pmod{p}$. Thus, $p \mid \binom{2k}{k} \binom{4k}{2k}$ for $\frac{p}{4} < k < p$. Now combining Lemma 2.2, Theorem 2.1 with Corollary 4.1 gives the result.

Theorem 4.3. *Let $p \equiv 1, 3 \pmod{8}$ be a prime and so $p = c^2 + 2d^2$ with $c, d \in \mathbb{Z}$ and $c \equiv 1 \pmod{4}$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} \left(2c - \frac{p}{2c} \right) \pmod{p^2}.$$

Proof. By Lemma 2.2, Theorem 2.1 and [BE, Theorems 5.12 and 5.17],

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} &\equiv \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \equiv P_{[\frac{p}{4}]}(0) \equiv - \sum_{x=0}^{p-1} \left(\frac{x^3 + 4x^2 + 2x}{p} \right) \\ &= - \sum_{x=0}^{p-1} \left(\frac{(-x)^3 + 4(-x)^2 + 2(-x)}{p} \right) = - \left(\frac{-1}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 - 4x^2 + 2x}{p} \right) \\ &\equiv (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c \pmod{p}. \end{aligned}$$

Set $\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} = (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp$. Then

$$\left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 = \left((-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 2c + qp \right)^2 \equiv 4c^2 + (-1)^{[\frac{p}{8}] + \frac{p-1}{2}} 4cqp \pmod{p^2}.$$

Taking $x = \frac{1}{128}$ in Theorem 4.1 we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \pmod{p^2}.$$

From [M] and [Su4] we have $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} / 256^k \equiv 4c^2 - 2p \pmod{p^2}$. Thus

$$4c^2 - 2p \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{128^k} \right)^2 \equiv 4c^2 + (-1)^{\lfloor \frac{p}{8} \rfloor + \frac{p-1}{2}} 4cqp \pmod{p^2}$$

and hence $q \equiv -(-1)^{\lfloor \frac{p}{8} \rfloor + \frac{p-1}{2}} \frac{1}{2c} \pmod{p}$. So the theorem is proved.

We note that Theorem 4.3 was conjectured by Zhi-Wei Sun in [Su1].

5. Congruences for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} / m^k$.

Theorem 5.1. *Let $p \neq 2, 3, 7$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{648^k} &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}, \\ 4a^2 \pmod{p} & \text{if } p = a^2 + 4b^2 \equiv 1 \pmod{4}, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-144)^k} &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}, \\ 4A^2 \pmod{p} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3}, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-3969)^k} &\equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}, \\ 4C^2 \pmod{p} & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}. \end{cases} \end{aligned}$$

Proof. Taking $m = 648, -144, -3969$ in Theorem 4.2 and then applying Theorems 2.2-2.4 and (1.6) we deduce the result.

We mention that Theorem 5.1 was conjectured by the author in [S2].

Lemma 5.1 ([S3, Lemma 4.1]). *Let p be an odd prime and let a, m, n be algebraic numbers which are integral for p . Then*

$$\sum_{x=0}^{p-1} (x^3 + a^2mx + a^3n)^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}} \sum_{x=0}^{p-1} (x^3 + mx + n)^{\frac{p-1}{2}} \pmod{p}.$$

Moreover, if a, m, n are congruent to rational integers modulo p , then

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + a^2mx + a^3n}{p} \right) = \left(\frac{a}{p} \right) \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right).$$

Theorem 5.2. *Let $p \neq 2, 3, 7$ be a prime. Then*

$$P_{[\frac{p}{4}]} \left(\frac{5\sqrt{-7}}{9} \right) \equiv \begin{cases} \left(\frac{3(7+\sqrt{-7})}{p} \right) \left(\frac{C}{7} \right) 2C \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p} & \text{if } p \equiv 3, 5, 6 \pmod{7} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{81^k} \equiv \begin{cases} 4C^2 \pmod{p} & \text{if } p \equiv 1, 2, 4 \pmod{7} \text{ and so } p = C^2 + 7D^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

Proof. By Theorem 2.1,

$$P_{[\frac{p}{4}]} \left(\frac{5\sqrt{-7}}{9} \right) \equiv - \left(\frac{6}{p} \right) \sum_{x=0}^{p-1} \left(x^3 - \frac{5}{2}(3 + \sqrt{-7})x + 7 + 5\sqrt{-7} \right)^{\frac{p-1}{2}} \pmod{p}.$$

Since

$$\frac{-\frac{5}{2}(3 + \sqrt{-7})}{-35} = \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^2 \quad \text{and} \quad \frac{7 + 5\sqrt{-7}}{-98} = \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^3,$$

by the above and Lemma 5.1 we have

$$P_{[\frac{p}{4}]} \left(\frac{5\sqrt{-7}}{9} \right) \equiv - \left(\frac{6}{p} \right) \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^{\frac{p-1}{2}} \sum_{x=0}^{p-1} \left(\frac{x^3 - 35x - 98}{p} \right) \pmod{p}.$$

As $x^3 + 21x^2 + 112x = (x + 7)^3 - 35(x + 7) - 98$, by (2.4) we get

$$(5.1) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 - 35x - 98}{p} \right) = \begin{cases} -2C \left(\frac{C}{7} \right) & \text{if } p = C^2 + 7D^2 \equiv 1, 2, 4 \pmod{7}, \\ 0 & \text{if } p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

For $p \equiv 1, 2, 4 \pmod{7}$ we see that

$$\left(\frac{6}{p} \right) \left(\frac{1 - \sqrt{-7}}{2\sqrt{-7}} \right)^{\frac{p-1}{2}} = \left(\frac{6}{p} \right) \left(\frac{7 + \sqrt{-7}}{2 \cdot (-7)} \right)^{\frac{p-1}{2}} \equiv \left(\frac{3}{p} \right) \left(\frac{7 + \sqrt{-7}}{p} \right) \pmod{p}.$$

Thus, from the above we deduce the congruence for $P_{[\frac{p}{4}]} \left(\frac{5\sqrt{-7}}{9} \right) \pmod{p}$. Applying Theorem 4.2 (with $m = 81$) we obtain the remaining result.

Let $p > 3$ be a prime and let \mathbb{F}_p be the field of p elements. For $m, n \in \mathbb{F}_p$ let $\#E_p(x^3 + mx + n)$ be the number of points on the curve $E: y^2 = x^3 + mx + n$ over the field \mathbb{F}_p . It is well known that (see for example [S1, pp.221-222])

$$(5.2) \quad \#E_p(x^3 + mx + n) = p + 1 + \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right).$$

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field and the curve $y^2 = x^3 + mx + n$ has complex multiplication by an order in K . By Deuring's theorem ([C, Theorem 14.16],[PV],[I]), we have

$$(5.3) \quad \#E_p(x^3 + mx + n) = \begin{cases} p + 1 & \text{if } p \text{ is inert in } K, \\ p + 1 - \pi - \bar{\pi} & \text{if } p = \pi\bar{\pi} \text{ in } K, \end{cases}$$

where π is in an order in K and $\bar{\pi}$ is the conjugate number of π . If $4p = u^2 + dv^2$ with $u, v \in \mathbb{Z}$, we may take $\pi = \frac{1}{2}(u + v\sqrt{-d})$. Thus,

$$(5.4) \quad \sum_{x=0}^{p-1} \left(\frac{x^3 + mx + n}{p} \right) = \begin{cases} \pm u & \text{if } 4p = u^2 + dv^2 \text{ with } u, v \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

In [JM] and [PV] the sign of u in (5.4) was determined for those imaginary quadratic fields K with class number 1. In [LM] and [I] the sign of u in (5.4) was determined for imaginary quadratic fields K with class number 2.

Theorem 5.3. *Let p be a prime such that $p \equiv \pm 1 \pmod{12}$. Then*

$$P_{\left[\frac{p}{4}\right]} \left(\frac{7}{12}\sqrt{3} \right) \equiv \begin{cases} \left(\frac{2+2\sqrt{3}}{p} \right) 2x \pmod{p} & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12} \text{ with } 3 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 11 \pmod{12} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-12288)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12}, \\ 0 \pmod{p^2} & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Proof. From [I, p.133] we know that the elliptic curve defined by the equation $y^2 = x^3 - (120 + 42\sqrt{3})x + 448 + 336\sqrt{3}$ has complex multiplication by the order of discriminant -36 . Thus, by (5.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 - (120 + 42\sqrt{3})n + 448 + 336\sqrt{3}}{p} \right) \\ &= \begin{cases} -2x \left(\frac{1+\sqrt{3}}{p} \right) & \text{if } p = x^2 + 9y^2 \equiv 1 \pmod{12} \text{ with } 3 \mid x - 1, \\ 0 & \text{if } p \equiv 11 \pmod{12}. \end{cases} \end{aligned}$$

By Theorem 2.1,

$$\begin{aligned} P_{\left[\frac{p}{4}\right]} \left(\frac{7}{12}\sqrt{3} \right) &\equiv - \left(\frac{6}{p} \right) \sum_{n=0}^{p-1} \left(n^3 - \frac{60 + 21\sqrt{3}}{8}n + \frac{28 + 21\sqrt{3}}{4} \right)^{\frac{p-1}{2}} \\ &\equiv - \left(\frac{6}{p} \right) \sum_{n=0}^{p-1} \left(\left(\frac{n}{4} \right)^3 - \frac{60 + 21\sqrt{3}}{8} \cdot \frac{n}{4} + \frac{28 + 21\sqrt{3}}{4} \right)^{\frac{p-1}{2}} \\ &\equiv - \left(\frac{6}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 - (120 + 42\sqrt{3})n + 448 + 336\sqrt{3}}{p} \right) \pmod{p}. \end{aligned}$$

Now combining all the above we obtain the congruence for $P_{[\frac{p}{4}]}(\frac{7}{12}\sqrt{3}) \pmod{p}$. Applying Theorem 4.2 (with $m = -12288$ and $t = \frac{7}{12}\sqrt{3}$) we deduce the remaining result.

Remark 5.1 In [Su1, Conjecture A24], Z.W. Sun conjectured that for any prime $p > 3$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-12288)^k} \equiv \begin{cases} (-1)^{[\frac{x}{6}]}(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ and } 4 \mid x - 1, \\ -4(\frac{xy}{3})xy \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{12} \text{ and } 4 \mid x - 1, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Theorem 5.4. *Let p be an odd prime such that $p \equiv \pm 1 \pmod{8}$. Then*

$$P_{[\frac{p}{4}]}(\frac{2\sqrt{2}}{3}) \equiv \begin{cases} (-1)^{\frac{p-1}{2}}(\frac{\sqrt{2}}{p})(\frac{x}{3})2x \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{48^{2k}} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p^2} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases}$$

Proof. From [I, p.133] we know that the elliptic curve defined by the equation $y^2 = x^3 + (-21 + 12\sqrt{2})x - 28 + 22\sqrt{2}$ has complex multiplication by the order of discriminant -24 . Thus, by (5.4) and [I, Theorem 3.1] we have

$$\begin{aligned} & \sum_{n=0}^{p-1} \left(\frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right) \\ &= \begin{cases} 2x(\frac{2x}{3})(\frac{1+\sqrt{2}}{p}) & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 0 & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

By Theorem 2.1,

$$P_{[\frac{p}{4}]}(\frac{2\sqrt{2}}{3}) \equiv -\left(\frac{6}{p}\right) \sum_{n=0}^{p-1} \left(n^3 - \frac{15 + 6\sqrt{2}}{2}n + 7 + 6\sqrt{2} \right)^{\frac{p-1}{2}} \pmod{p}.$$

Since

$$\frac{-(15 + 6\sqrt{2})/2}{-21 + 12\sqrt{2}} = \left(\frac{\sqrt{2} + 1}{\sqrt{2}} \right)^2 \quad \text{and} \quad \frac{7 + 6\sqrt{2}}{-28 + 22\sqrt{2}} = \left(\frac{\sqrt{2} + 1}{\sqrt{2}} \right)^3,$$

by Lemma 5.1 and the above we have

$$\begin{aligned} P_{[\frac{p}{4}]}(\frac{2\sqrt{2}}{3}) &\equiv -\left(\frac{6}{p}\right) \left(\frac{\sqrt{2}(\sqrt{2} + 1)}{p} \right) \sum_{n=0}^{p-1} \left(\frac{n^3 + (-21 + 12\sqrt{2})n - 28 + 22\sqrt{2}}{p} \right) \\ &= \begin{cases} -\left(\frac{6}{p}\right) \left(\frac{\sqrt{2}}{p} \right) 2x \left(\frac{2x}{3} \right) \pmod{p} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{24}. \end{cases} \end{aligned}$$

This yields the result for $P_{[\frac{p}{4}]}(\frac{2\sqrt{2}}{3}) \pmod{p}$. Taking $m = 48^2$ and $t = \frac{2}{3}\sqrt{2}$ in Theorem 4.2 and applying the above we deduce the remaining result.

Let $b \in \{3, 5, 11, 29\}$ and $f(b) = 48^2, 12^4, 1584^2, 396^4$ according as $b = 3, 5, 11, 29$. For any odd prime p with $p \nmid bf(b)$, Z.W. Sun conjectured that ([Su1, Conjectures A14, A16, A18 and A21])

$$(5.5) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{f(b)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2by^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } p = 2x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-2b}{p}) = -1. \end{cases}$$

For $m \in \{5, 13, 37\}$ let

$$(5.6) \quad f(m) = \begin{cases} -1024 & \text{if } m = 5, \\ -82944 & \text{if } m = 13, \\ -2^{10} \cdot 21^4 & \text{if } m = 37 \end{cases} \quad \text{and} \quad t(m) = \sqrt{1 - \frac{256}{f(m)}} = \begin{cases} \frac{\sqrt{5}}{2} & \text{if } m = 5, \\ \frac{5}{18}\sqrt{13} & \text{if } m = 13, \\ \frac{145}{882}\sqrt{37} & \text{if } m = 37. \end{cases}$$

Suppose that p is an odd prime such that $p \nmid mf(m)$. In [Su1], Z.W. Sun conjectured that

$$(5.7) \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{f(m)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{m}{p}) = (\frac{-1}{p}) = 1 \text{ and so } p = x^2 + my^2, \\ 2p - 2x^2 \pmod{p^2} & \text{if } (\frac{m}{p}) = (\frac{-1}{p}) = -1 \text{ and so } 2p = x^2 + my^2, \\ 0 \pmod{p^2} & \text{if } (\frac{m}{p}) = -(\frac{-1}{p}). \end{cases}$$

Theorem 5.5. *Let $m \in \{5, 13, 37\}$ and let p be an odd prime such that $(\frac{m}{p}) = 1$ and $p \nmid f(m)$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{f(m)^k} \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + my^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. Let $t(m)$ be given in (5.6). From [LM, Table II] we know that the elliptic curve defined by the equation $y^2 = x^3 + 4x^2 + (2 - 2t(m))x$ has complex multiplication by the order of discriminant $-4m$. Thus, by (5.4) we have

$$\sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - 2t(m))n}{p} \right) = \begin{cases} 2x & \text{if } p \equiv 1 \pmod{4} \text{ and so } p = x^2 + my^2, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

for $m = 5$ see also [LM, Theorem 11]. Now applying the above and Theorem 4.2 we deduce the result.

Remark 5.2 Let $p \equiv 1 \pmod{20}$ be a prime and hence $p = a^2 + 4b^2 = x^2 + 5y^2$ with $a, b, x, y \in \mathbb{Z}$. A result of Cauchy ([BEW, p. 291]) states that

$$\left(\frac{p-1}{2} \right)^2 \equiv \begin{cases} 4x^2 \pmod{p} & \text{if } 5 \nmid a, \\ -4x^2 \pmod{p} & \text{if } 5 \mid a. \end{cases}$$

Using the arguments similar to the proof of Theorem 5.5, [LM, Table II] (or [I]) and (5.4) we see that for any odd prime p ,

$$(5.8) \quad \begin{aligned} \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{8}{9}\sqrt{5})n}{p} \right) &= \begin{cases} 2x & \text{if } (\frac{5}{p}) = (\frac{-2}{p}) = 1 \text{ and so } p = x^2 + 10y^2, \\ 0 & \text{if } (\frac{5}{p}) = 1 \text{ and } (\frac{-2}{p}) = -1, \end{cases} \\ \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{140}{99}\sqrt{2})n}{p} \right) &= \begin{cases} 2x & \text{if } (\frac{2}{p}) = (\frac{-11}{p}) = 1 \text{ and so } p = x^2 + 22y^2, \\ 0 & \text{if } (\frac{2}{p}) = 1 \text{ and } (\frac{-11}{p}) = -1, \end{cases} \\ \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{3640}{9801}\sqrt{29})n}{p} \right) &= \begin{cases} 2x & \text{if } (\frac{-2}{p}) = (\frac{29}{p}) = 1 \text{ and so } p = x^2 + 58y^2, \\ 0 & \text{if } (\frac{-2}{p}) = -1 \text{ and } (\frac{29}{p}) = 1, \end{cases} \\ \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{40}{49}\sqrt{6})n}{p} \right) &= \begin{cases} 2x & \text{if } p \equiv 1, 19 \pmod{24} \text{ and so } p = x^2 + 18y^2, \\ 0 & \text{if } p \equiv 5, 23 \pmod{24}, \end{cases} \\ \sum_{n=0}^{p-1} \left(\frac{n^3 + 4n^2 + (2 - \frac{161}{180}\sqrt{5})n}{p} \right) &= \begin{cases} 2x & \text{if } p \equiv 1, 9 \pmod{20} \text{ and so } p = x^2 + 25y^2, \\ 0 & \text{if } p \equiv 11, 19 \pmod{20}. \end{cases} \end{aligned}$$

From (5.8) and Theorem 4.2 we deduce the following results:

$$(5.9) \quad \begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{12^{4k}} &\equiv \begin{cases} 4x^2 \pmod{p} & \text{if } (\frac{5}{p}) = (\frac{-2}{p}) = 1 \text{ and so } p = x^2 + 10y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{5}{p}) = 1 \text{ and } (\frac{-2}{p}) = -1, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{1584^{2k}} &\equiv \begin{cases} 4x^2 \pmod{p} & \text{if } (\frac{2}{p}) = (\frac{-11}{p}) = 1 \text{ and so } p = x^2 + 22y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{2}{p}) = 1 \text{ and } (\frac{-11}{p}) = -1, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{396^{4k}} &\equiv \begin{cases} 4x^2 \pmod{p} & \text{if } (\frac{-2}{p}) = (\frac{29}{p}) = 1 \text{ and so } p = x^2 + 58y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-2}{p}) = -1 \text{ and } (\frac{29}{p}) = 1, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{28^{4k}} &\equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1, 19 \pmod{24} \text{ and so } p = x^2 + 18y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 23 \pmod{24}, \end{cases} \\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{14} \cdot 3^4 \cdot 5)^k} &\equiv \begin{cases} 4x^2 \pmod{p} & \text{if } p \equiv 1, 9 \pmod{20} \text{ and so } p = x^2 + 25y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 19 \pmod{20}. \end{cases} \end{aligned}$$

We remark that all the congruences in (5.9) were conjectured by Zhi-Wei Sun in [Su1].

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