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Super congruences concerning Bernoulli polynomials

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Let $p>3$ be a prime, and let a be a rational p-adic integer. Let $\{B_n(x)\}$ denote the Bernoulli polynomials given by $B_0=1$, $\sum_{k=0}^{n-1}\binom nkB_k=0\ (n\ge 2)$ and $B_n(x)=\sum_{k=0}^n\binom nkB_kx^{n-k}\ (n\ge 0)$. In this paper, using Bernoulli polynomials we establish congruences for $\sum_{k=1}^{p-1}\frac 1k\binom ak\binom{-1-a}k \pmod{p^3}$ and $\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}k\binom ak \pmod{p^2}$. As a consequence we solve the following conjecture of Z.W. Sun:

$$\sum_{k=1}^{p-1}\frac{\binom{2k}k\binom{4k}{2k}}{64^kk}\equiv -3H_{\frac{p-1}2}+\frac 74p^2B_{p-3}\pmod{p^3},$$

where $H_n=1+\frac 12+\cdots+\frac 1n$.

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1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\ \sum_{k=0}^{n-1}\binom nkB_k=0\ (n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^n\binom nkB_kx^{n-k}\ (n\ge 0).$$

The Euler numbers $\{E_n\}$ are defined by $E_0=1$ and $E_n=-\sum_{k=1}^{[n/2]}\binom n{2k}E_{n-2k}\ (n\ge 1)$, where $[a]$ is the greatest integer not exceeding a. In [S5] the author introduced the sequence $\{U_n\}$ given by $U_0=1$ and $U_n=-2\sum_{k=1}^{[n/2]}\binom n{2k}U_{n-2k}\ (n\ge 1)$. It is well known that $B_{2n+1}=0$ and $E_{2n-1}=U_{2n-1}=0$ for any positive integer n. $\{B_n\}$, $\{E_n\}$ and $\{U_n\}$ are important sequences and they have many interesting properties and applications. See [B], [MOS] and [S1-S6].

Let $\mathbb{Z}$ be the set of integers. For a prime p let $\mathbb{Z}_p$ denote the set of rational $p-adic$ integers. For a $p-adic$ integer a let $\langle a\rangle_p \in \{0,1,\ldots,p-1\}$ be given by $a \equiv \langle a\rangle_p \pmod p$. In [S7] the author showed that for any odd prime p and $a\in\mathbb{Z}_p$,

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k} \equiv (-1)^{\langle a\rangle_p} \pmod{p^2}. \tag{1.1}$$

As pointed out in [S7], we have

$$\begin{aligned}\binom{-\frac12}{k}^2 &= \frac{\binom{2k}{k}^2}{16^k},\quad \binom{-\frac13}{k}\binom{-\frac23}{k} = \frac{\binom{2k}{k}\binom{3k}{k}}{27^k},\\ \binom{-\frac14}{k}\binom{-\frac34}{k} &= \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k},\quad \binom{-\frac16}{k}\binom{-\frac56}{k} = \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}.\end{aligned} \tag{1.2}$$

Let $p>3$ be a prime and $a\in\mathbb{Z}_p$. In [T2] Tauraso obtained a congruence for $\sum_{k=1}^{p-1}\frac1k\binom{a}{k}\binom{-1-a}{k} \pmod{p^2}$. In [Su], the author's brother Z.W. Sun conjectured that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k k} \equiv -3H_{\frac{p-1}{2}} + \frac74 p^2 B_{p-3} \pmod{p^3}, \tag{1.3}$$

where $H_n = 1+\frac12+\cdots+\frac1n$.

Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $t=(a-\langle a\rangle_p)/p$. In this paper we prove that

$$\sum_{k=1}^{p-1}\frac{\binom{a}{k}\binom{-1-a}{k}}{k} \equiv -\frac23 p^2 t(t+1)B_{p-3}(-a) - 2\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^3}.$$

As consequences we completely determine

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k k},\quad \sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k k},\quad \sum_{k=1}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k k} \pmod{p^3}.$$

In particular, we confirm (1.3) and prove that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k k} \equiv 3q_p(3) - \frac32 pq_p(3)^2 + p^2\Big(q_p(3)^3 + \frac{52}{27}B_{p-3}\Big) \pmod{p^3}, \tag{1.4}$$

where $q_p(a) = (a^{p-1}-1)/p$. We also show that

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{a}{k} \equiv \frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)} - \frac{a-\langle a\rangle_p}{2}B_{p-2}(-a) \pmod{p^2}$$

and completely determine $\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{a}{k} \pmod{p^2}$ in the cases $a=-\frac13,-\frac14,-\frac16$. For example,

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/4}{k} \equiv -3q_p(2) + p\Big(\frac32 q_p(2)^2 + \Big(2-\Big(\frac{-1}{p}\Big)\Big)E_{p-3}\Big) \pmod{p^2}, \tag{1.5}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol.

2. Congruences for $\sum_{k=1}^{p-1}\frac{1}{k}\binom{a}{k}\binom{-1-a}{k} \pmod{p^3}$

For any positive integer n and variable a let

$$S_n(a)=\sum_{k=1}^{n}\frac{1}{k}\binom{a}{k}\binom{-1-a}{k}.$$

Then

$$\begin{aligned}S_n(a)-S_n(a-1)&=\sum_{k=1}^{n}\frac{1}{k}\Big\{\binom{a}{k}\binom{-1-a}{k}-\binom{a-1}{k}\binom{-a}{k}\Big\}\\&=\sum_{k=1}^{n}\frac{1}{k}\binom{a}{k}\binom{-a}{k}\Big(\frac{a+k}{a}-\frac{a-k}{a}\Big)=\frac{2}{a}\sum_{k=1}^{n}\binom{a}{k}\binom{-a}{k}.\end{aligned}$$

By [S7, (4.5)] or induction on n,

$$\sum_{k=0}^{n}\binom{a}{k}\binom{-a}{k}=\binom{n+a}{n}\binom{n-a}{n}=\binom{a-1}{n}\binom{-a-1}{n}.$$

Thus,

$$S_n(a)-S_n(a-1)=\frac{2}{a}\binom{a-1}{n}\binom{-a-1}{n}-\frac{2}{a}. \tag{2.1}$$

Lemma 2.1 ([S7, Lemma 4.2]). *Let p be an odd prime, $m\in\{1,2,\ldots,p-1\}$ and $t\in\mathbb{Z}_p$. Then*

$$\binom{m+pt-1}{p-1}\equiv\frac{pt}{m}-\frac{p^2t^2}{m^2}+\frac{p^2t}{m}H_m \pmod{p^3}.$$

By (2.1) and Lemma 2.1, if p is an odd prime and $a\in\mathbb{Z}_p$ with $a\not\equiv 0 \pmod p$, then

$$S_{p-1}(a)-S_{p-1}(a-1)\equiv\frac{2t(t+1)}{a\langle a\rangle_p^2}p^2-\frac{2}{a}\equiv\frac{2t(t+1)}{\langle a\rangle_p^3}p^2-\frac{2}{a} \pmod{p^3}, \tag{2.2}$$

where $t=(a-\langle a\rangle_p)/p$.

Lemma 2.2. *Let $p>3$ be a prime and $t\in\mathbb{Z}_p$. Then*

$$\sum_{k=1}^{p-1}\frac{1}{k}\binom{pt}{k}\binom{-1-pt}{k}\equiv-\frac{2}{3}p^2tB_{p-3} \pmod{p^3}.$$

Proof. For $k\in\{1,2,\ldots,p-1\}$ we see that

$$\begin{aligned}\binom{pt}{k}\binom{-1-pt}{k}&=\frac{pt(pt-1)\cdots(pt-k+1)(-1-pt)(-2-pt)\cdots(-k-pt)}{k!^2}\\&=\frac{(-1)^kpt(pt+k)}{k!^2}(p^2t^2-1^2)\cdots(p^2t^2-(k-1)^2)\end{aligned}$$

$$\equiv -\frac{pt(pt+k)}{k^2} = -\frac{p^2t^2}{k^2} - \frac{pt}{k} \pmod{p^3}.$$

Thus,

$$\sum_{k=1}^{p-1} \frac{1}{k}\binom{pt}{k}\binom{-1-pt}{k} \equiv -p^2t^2\sum_{k=1}^{p-1}\frac{1}{k^3} - pt\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^3}.$$

By [L] or [S2, Corollary 5.1], $\sum_{k=1}^{p-1}\frac{1}{k^3} \equiv 0 \pmod{p}$ and $\sum_{k=1}^{p-1}\frac{1}{k^2} \equiv \frac{2}{3}pB_{p-3} \pmod{p^2}$. Thus the result follows.

Lemma 2.3. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod{p}$ and $t = (a - \langle a\rangle_p)/p$. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{a}{k}\binom{-1-a}{k}}{k} \equiv -\frac{2}{3}p^2tB_{p-3} - 2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r} + 2pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2} + 2p^2t\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^3} \pmod{p^3}.$$

Proof. For $1 \le k \le \langle a\rangle_p$ we have $\langle a-k+1\rangle_p = \langle a\rangle_p - k+1$ and so $a-k+1 = \langle a-k+1\rangle_p + pt$. Using (2.2) we see that

$$\begin{aligned}
&S_{p-1}(a) - S_{p-1}(a-\langle a\rangle_p)\\
&= \sum_{k=1}^{\langle a\rangle_p}(S_{p-1}(a-k+1) - S_{p-1}(a-k)) \equiv \sum_{k=1}^{\langle a\rangle_p}\Big(\frac{2t(t+1)p^2}{\langle a-k+1\rangle_p^3} - \frac{2}{a-k+1}\Big)\\
&= 2t(t+1)p^2\sum_{k=1}^{\langle a\rangle_p}\frac{1}{(\langle a\rangle_p - k+1)^3} - 2\sum_{k=1}^{\langle a\rangle_p}\frac{1}{\langle a\rangle_p - k+1+pt}\\
&= 2t(t+1)p^2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^3} - 2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r+pt} \pmod{p^3}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r+pt} &= \sum_{r=1}^{\langle a\rangle_p}\frac{r^2-ptr+p^2t^2}{r^3-(pt)^3} \equiv \sum_{r=1}^{\langle a\rangle_p}\frac{r^2-ptr+p^2t^2}{r^3}\\
&= \sum_{r=1}^{\langle a\rangle_p}\frac{1}{r} - pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2} + p^2t^2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^3} \pmod{p^3}.
\end{aligned}$$

We then obtain

$$S_{p-1}(a) - S_{p-1}(a-\langle a\rangle_p) \equiv -2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r} + 2pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2} + 2tp^2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^3} \pmod{p^3}.$$

By Lemma 2.2, $S_{p-1}(a-\langle a\rangle_p) = S_{p-1}(pt) \equiv -\frac{2}{3}p^2tB_{p-3} \pmod{p^3}$. Thus the result follows.

Theorem 2.1. *Let $p > 3$ be a prime, $a \in \mathbb{Z}_p$, $a \not\equiv 0 \pmod{p}$ and $t = (a - \langle a\rangle_p)/p$. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{a}{k}\binom{-1-a}{k}}{k} \equiv -\frac{2}{3}p^2t(t+1)B_{p-3}(-a) - 2\frac{B_{p^2(p-1)}(-a) - B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^3}.$$

Proof. It is well known that (see [MOS])

$$\sum_{r=1}^{m} r^k = \frac{B_{k+1}(m+1)-B_{k+1}}{k+1}, \tag{2.3}$$

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}y^kB_{n-k}(x),\ B_n(1-x)=(-1)^nB_n(x). \tag{2.4}$$

Thus, using Euler's theorem we see that

$$\begin{aligned}
&pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}-\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}\\
&\equiv pt\sum_{r=1}^{\langle a\rangle_p}r^{p^2(p-1)-2}-\sum_{r=1}^{\langle a\rangle_p}r^{p^2(p-1)-1}\\
&=pt\frac{B_{p^2(p-1)-1}(\langle a\rangle_p+1)-B_{p^2(p-1)-1}}{p^2(p-1)-1}-\frac{B_{p^2(p-1)}(\langle a\rangle_p+1)-B_{p^2(p-1)}}{p^2(p-1)}\\
&=-pt\frac{B_{p^2(p-1)-1}(-\langle a\rangle_p)}{p^2(p-1)-1}-\frac{B_{p^2(p-1)}(-\langle a\rangle_p)-B_{p^2(p-1)}}{p^2(p-1)}\\
&=-pt\frac{B_{p^2(p-1)-1}(pt-a)}{p^2(p-1)-1}-\frac{B_{p^2(p-1)}(pt-a)-B_{p^2(p-1)}(-a)+B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)}\\
&=\frac{-pt}{p^2(p-1)-1}\sum_{k=0}^{p^2(p-1)-1}\binom{p^2(p-1)-1}{k}(pt)^kB_{p^2(p-1)-1-k}(-a)\\
&\quad-\frac{1}{p^2(p-1)}\sum_{k=1}^{p^2(p-1)}\binom{p^2(p-1)}{k}(pt)^kB_{p^2(p-1)-k}(-a)-\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^3}.
\end{aligned}$$

By [S1, Lemma 2.3], $B_m(-a)\in\mathbb{Z}_p$ for $m\not\equiv 0 \pmod{p-1}$ and $pB_m(-a)\in\mathbb{Z}_p$ for $m\equiv 0 \pmod{p-1}$. Thus,

$$\begin{aligned}
&pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}-\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}+\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)}\\
&\equiv pt\big(B_{p^2(p-1)-1}(-a)+(p^2(p-1)-1)ptB_{p^2(p-1)-2}(-a)\big)\\
&\qquad-ptB_{p^2(p-1)-1}(-a)-\frac{p^2(p-1)-1}{2}(pt)^2B_{p^2(p-1)-2}(-a)\\
&\equiv -p^2t^2B_{p^2(p-1)-2}(-a)+\frac{1}{2}p^2t^2B_{p^2(p-1)-2}(-a)\\
&=-\frac{1}{2}p^2t^2B_{(p^2-1)(p-1)+p-3}(-a) \pmod{p^3}.
\end{aligned}$$

By [S2, Corollary 3.1],

$$B_{(p^2-1)(p-1)+p-3}(-a)\equiv((p^2-1)(p-1)+p-3)\frac{B_{p-3}(-a)}{p-3}\equiv\frac{2}{3}B_{p-3}(-a) \pmod{p}.$$

Thus,

$$\text{(2.5)}\qquad pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}-\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}\equiv -\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)}-\frac{1}{3}p^2t^2B_{p-3}(-a) \pmod{p^3}.$$

By [S2, Lemma 3.2],

$$\text{(2.6)}\qquad \sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^3}\equiv\sum_{r=1}^{\langle a\rangle_p}r^{p-4}\equiv\frac{B_{p-3}(-a)-B_{p-3}}{p-3}\equiv -\frac{1}{3}\big(B_{p-3}(-a)-B_{p-3}\big) \pmod{p}.$$

Thus, from Lemma 2.3, (2.5) and (2.6) we derive that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{\binom{a}{k}\binom{-1-a}{k}}{k}&\equiv -\frac{2}{3}p^2tB_{p-3}-2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}+2pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}+2tp^2\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^3}\\ &\equiv -\frac{2}{3}p^2tB_{p-3}-2\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)}\\ &\quad-\frac{2}{3}p^2t^2B_{p-3}(-a)-\frac{2}{3}p^2t\big(B_{p-3}(-a)-B_{p-3}\big)\\ &=-\frac{2}{3}p^2t(t+1)B_{p-3}(-a)-2\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)} \pmod{p^3}.\end{aligned}$$

This completes the proof.

Lemma 2.4 ([MOS]). *For any positive integer n we have*

$$\begin{aligned}&B_{2n}\Big(\frac{1}{2}\Big)=(2^{1-2n}-1)B_{2n},\ B_{2n}\Big(\frac{1}{3}\Big)=\frac{3-3^{2n}}{2\cdot 3^{2n}}B_{2n},\\ &B_{2n}\Big(\frac{1}{4}\Big)=\frac{2-2^{2n}}{4^{2n}}B_{2n},\ B_{2n}\Big(\frac{1}{6}\Big)=\frac{(2-2^{2n})(3-3^{2n})}{2\cdot 6^{2n}}B_{2n}.\end{aligned}$$

Theorem 2.2. *Let $p>3$ be a prime. Then*

$$\text{(i)}\qquad \sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^kk}\equiv 3q_p(3)-\frac{3}{2}pq_p(3)^2+p^2\Big(q_p(3)^3+\frac{52}{27}B_{p-3}\Big) \pmod{p^3},$$

$$\text{(ii)}\qquad \sum_{k=1}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^kk}\equiv 6q_p(2)-3pq_p(2)^2+p^2\Big(2q_p(2)^3+\frac{7}{2}B_{p-3}\Big) \pmod{p^3},$$

$$\begin{aligned}\text{(iii)}\qquad \sum_{k=1}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^kk}&\equiv 4q_p(2)+3q_p(3)-p\Big(2q_p(2)^2+\frac{3}{2}q_p(3)^2\Big)\\ &\quad+p^2\Big(\frac{4}{3}q_p(2)^3+q_p(3)^3+\frac{455}{54}B_{p-3}\Big) \pmod{p^3}.\end{aligned}$$

Proof. By Lemma 2.4,

$$B_{p-3}\Big(\frac{1}{3}\Big)=\frac{3-3^{p-3}}{2\cdot 3^{p-3}}B_{p-3}\equiv 13B_{p-3} \pmod{p},$$

$$B_{p-3}\Big(\frac{1}{4}\Big) = \frac{2-2^{p-3}}{4^{p-3}} B_{p-3} \equiv 28B_{p-3} \pmod p,$$
$$B_{p-3}\Big(\frac{1}{6}\Big) = \frac{(2-2^{p-3})(3-3^{p-3})}{2\cdot 6^{p-3}} B_{p-3} \equiv 91B_{p-3} \pmod p.$$

By [S4, p.287],

$$\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{3})}{p^2(p-1)} \equiv \frac{3}{2}\big(q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3\big) \pmod{p^3}, \tag{2.7}$$

$$\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} \equiv 3\big(q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3\big) \pmod{p^3}, \tag{2.8}$$

$$\begin{aligned}\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{6})}{p^2(p-1)} &\equiv 2\Big(q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3\Big) \\ &\quad + \frac{3}{2}\Big(q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3\Big) \pmod{p^3}.\end{aligned} \tag{2.9}$$

Now taking $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ in Theorem 2.1 and then applying (1.2) and the above we deduce the result.

Remark 2.1 Let $p > 3$ be a prime. In [T2] Tauraso showed that $\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k} \equiv -2H_{\frac{p-1}{2}} \pmod{p^3}$, which can be deduced from Theorem 2.1 (with $a = -\frac{1}{2}$) and the congruence ([S2, Theorem 5.2(c)])

$$H_{\frac{p-1}{2}} \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \pmod{p^3}. \tag{2.10}$$

By (2.10), Theorem 2.2(ii) is equivalent to Z.W. Sun's conjecture (1.3).

3. Congruences for $\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\binom{a}{k} \pmod{p^2}$

For given positive integer n and variables a and b with $b \notin \{0, 1, \dots, n-1\}$ define

$$f_n(a,b) = \sum_{k=1}^{n} \frac{\binom{a}{k}}{k\binom{b}{k}}.$$

Then

$$f_n(a,b) - f_n(a-1,b) = \sum_{k=1}^{n} \frac{\binom{a}{k} - \binom{a-1}{k}}{k\binom{b}{k}} = \sum_{k=1}^{n} \frac{\frac{1}{k}\binom{a-1}{k-1}}{\binom{b}{k}} = \frac{1}{a}\sum_{k=1}^{n} \frac{\binom{a}{k}}{\binom{b}{k}}.$$

By Lerch's theorem ([B, p.86]) or induction on n,

$$\sum_{k=0}^{n} \frac{\binom{a}{k}}{\binom{b}{k}} = \frac{b+1}{b+1-a}\Big\{1 - \frac{\binom{a}{n+1}}{\binom{b+1}{n+1}}\Big\}. \tag{3.1}$$

Thus,

$$
\begin{aligned}
f_n(a,b)-f_n(a-1,b) &= \frac{1}{a}\Big\{\frac{b+1}{b+1-a}-1-\frac{b+1}{b+1-a}\cdot\frac{\binom{a}{n+1}}{\binom{b+1}{n+1}}\Big\} \\
&= \frac{1}{b+1-a}-\frac{1}{b+1-a}\cdot\frac{\binom{a-1}{n}}{\binom{b}{n}}.
\end{aligned}
\tag{3.2}
$$

Lemma 3.1. *Let p be an odd prime, $n\in\{1,2,3,\dots\}$, $a,b\in\mathbb{Z}_p, 1\le\langle a\rangle_p\le n\le\langle b\rangle_p$ and $a=\langle a\rangle_p+pt$. Then*

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{\binom{a}{k}}{k\binom{b}{k}} &\equiv pt\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2\binom{b}{k}}+\sum_{r=1}^{\langle a\rangle_p}\frac{1}{b+1-r}+pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{(b+1-r)^2} \\
&\quad -\frac{pt}{\binom{b}{n}}\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^{n-r}}{(b+1-r)r\binom{n}{r}} \pmod{p^2}.
\end{aligned}
$$

Proof. By (3.2),

$$
\begin{aligned}
f_n(a,b)-f_n(a-\langle a\rangle_p,b) &= \sum_{k=1}^{\langle a\rangle_p}(f_n(a-k+1,b)-f_n(a-k,b)) \\
&= \sum_{k=1}^{\langle a\rangle_p}\frac{1}{b+1-(a-k+1)}\Big(1-\frac{\binom{a-k}{n}}{\binom{b}{n}}\Big) \\
&= \sum_{k=1}^{\langle a\rangle_p}\frac{1}{b+1-a+\langle a\rangle_p-(\langle a\rangle_p-k+1)}\Big(1-\frac{1}{\binom{b}{n}}\binom{\langle a\rangle_p-k+1+a-\langle a\rangle_p-1}{n}\Big).
\end{aligned}
$$

Substituting k with $\langle a\rangle_p+1-r$ in the above we obtain

$$
f_n(a,b)-f_n(pt,b)=\sum_{r=1}^{\langle a\rangle_p}\frac{1}{b+1-r-pt}-\frac{1}{\binom{b}{n}}\sum_{r=1}^{\langle a\rangle_p}\frac{1}{b+1-r-pt}\binom{r+pt-1}{n}. \tag{3.3}
$$

For $1\le r\le\langle a\rangle_p$ we see that

$$
\begin{aligned}
\binom{r-1+pt}{n} &= \frac{(r-1+pt)(r-2+pt)\cdots(1+pt)pt(pt-1)\cdots(pt-(n-r))}{n!} \\
&\equiv \frac{(r-1)!\cdot pt\cdot(-1)^{n-r}\cdot(n-r)!}{n!}=(-1)^{n-r}\frac{pt}{r\binom{n}{r}} \pmod{p^2}
\end{aligned}
$$

and so

$$
\begin{aligned}
f_n(a,b)-f_n(pt,b) &\equiv \sum_{r=1}^{\langle a\rangle_p}\frac{1}{b+1-r-pt}-\frac{1}{\binom{b}{n}}\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^{n-r}}{b+1-r-pt}\cdot\frac{pt}{r\binom{n}{r}} \\
&\equiv \sum_{r=1}^{\langle a\rangle_p}\frac{b+1-r+pt}{(b+1-r)^2}-\frac{pt}{\binom{b}{n}}\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^{n-r}}{(b+1-r)r\binom{n}{r}} \pmod{p^2}.
\end{aligned}
$$

On the other hand,

$$f_n(pt,b)=\sum_{k=1}^{n}\frac{\binom{pt}{k}}{k\binom{b}{k}}=\sum_{k=1}^{n}\frac{pt\binom{pt-1}{k-1}}{k^2\binom{b}{k}}\equiv pt\sum_{k=1}^{n}\frac{\binom{-1}{k-1}}{k^2\binom{b}{k}}=pt\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^2\binom{b}{k}} \pmod{p^2}.$$

Thus, the result follows.

Theorem 3.1. *Let $p>3$ be a prime, $a\in\mathbb{Z}_p$ and $a\not\equiv 0 \pmod p$. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{a}{k}\equiv\frac{B_{p^2(p-1)}(-a)-B_{p^2(p-1)}}{p^2(p-1)}-\frac{a-\langle a\rangle_p}{2}B_{p-2}(-a) \pmod{p^2}.$$

Proof. Set $a=\langle a\rangle_p+pt$. By [S2, Lemma 3.2],

$$\sum_{r=1}^{\langle a\rangle_p}r^{p-3}\equiv(-1)^{p-2}\frac{B_{p-2}(-a)-B_{p-2}}{p-2}\equiv\frac{1}{2}B_{p-2}(-a) \pmod p. \tag{3.4}$$

As $\binom{-1}{k}=(-1)^k$, taking $b=-1$ and $n=p-1$ in Lemma 3.1 and then applying (2.5), (3.4) and the known fact (see [S2, Corollary 5.1]) $\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0 \pmod p$ we see that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\binom{a}{k}&=\sum_{k=1}^{p-1}\frac{\binom{a}{k}}{k\binom{-1}{k}}\equiv -pt\sum_{k=1}^{p-1}\frac{1}{k^2}-\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r}+pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}+pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2}\\&\equiv\frac{B_{p^2(p-1)}-B_{p^2(p-1)}(-a)}{p^2(p-1)}+\frac{pt}{2}B_{p-2}(-a) \pmod{p^2}.\end{aligned}$$

This yields the result.

Remark 3.1 Let $p>3$ be a prime. Taking $a=-\frac{1}{2}$ in Theorem 3.1 and then applying Lemma 2.4 we deduce that

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/2}{k}\equiv -2q_p(2)+pq_p(2)^2 \pmod{p^2}.$$

In [T1], using a special method Tauraso proved the following stronger congruence:

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/2}{k}\equiv H_{\frac{p-1}{2}} \pmod{p^3}.$$

Theorem 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/4}{k}\equiv -3q_p(2)+p\Big(\frac{3}{2}q_p(2)^2+\Big(2-\Big(\frac{-1}{p}\Big)\Big)E_{p-3}\Big) \pmod{p^2}.$$

Proof. Taking $a=-\frac{1}{4}$ in Theorem 3.1 we see that

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/4}{k}\equiv\frac{B_{p^2(p-1)}(\frac{1}{4})-B_{p^2(p-1)}}{p^2(p-1)}-\frac{-\frac{1}{4}-\langle-\frac{1}{4}\rangle_p}{2}B_{p-2}\Big(\frac{1}{4}\Big) \pmod{p^2}.$$

By (2.8), $\frac{B_{p^2(p-1)}-B_{p^2(p-1)}(\frac 14)}{p^2(p-1)} \equiv 3q_p(2)-\frac 32pq_p(2)^2 \pmod{p^2}$. It is known (see for example [S4, Lemma 2.5]) that $E_{2n}=-4^{2n+1}\frac{B_{2n+1}(\frac 14)}{2n+1}$. Thus, $E_{p-3}=-4^{p-2}\frac{B_{p-2}(\frac 14)}{p-2}\equiv \frac{B_{p-2}(\frac 14)}{8} \pmod p$. Now, from the above we deduce the result.

Theorem 3.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/3}{k}\equiv -\frac 32q_p(3)+p\Big(\frac 34q_p(3)^2+\frac{3-(\frac{-3}{p})}{2}U_{p-3}\Big) \pmod{p^2}.$$

Proof. Taking $a=-\frac 13$ in Theorem 3.1 we see that

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/3}{k}\equiv \frac{B_{p^2(p-1)}(\frac 13)-B_{p^2(p-1)}}{p^2(p-1)}-\frac{-\frac 13-\langle -\frac 13\rangle_p}{2}B_{p-2}\Big(\frac 13\Big) \pmod{p^2}.$$

By (2.7), $\frac{B_{p^2(p-1)}-B_{p^2(p-1)}(\frac 13)}{p^2(p-1)}\equiv \frac 32q_p(3)-\frac 34pq_p(3)^2 \pmod{p^2}$. By [S5, p.217], $B_{p-2}(\frac 13)\equiv 6U_{p-3} \pmod p$. Now, from the above we deduce the result.

Theorem 3.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\binom{-1/6}{k}$$
$$\equiv -2q_p(2)-\frac 32q_p(3)+p\Big(q_p(2)^2+\frac 34q_p(3)^2+\frac 52\Big(3-2\Big(\frac{-3}{p}\Big)\Big)U_{p-3}\Big) \pmod{p^2}.$$

Proof. Taking $a=-\frac 16$ in Theorem 3.1 and then applying (2.9) and the fact (see [S5, p.216]) $B_{p-2}(\frac 16)\equiv 30U_{p-3} \pmod p$ we deduce the result.

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